PHILOSOPHICAL TRANSACTIONS.

I. *Memoir on the Integration of Partial Differential Equations of the Second Order in Three Independent Variables when an Intermediary Integral does not exist in general.*

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The general feature of most methods for the integration of partial differential equations in two independent variables is, in some form or other, the construction of a set of subsidiary equations in only a single independent variable; and this applies to all orders. In particular, for the first order in any number of variables (not merely in two), the subsidiary system is a set of ordinary equations in a single independent variable, containing as many equations as dependent variables to be determined by that subsidiary system. For equations of the second order which possess an intermediary integral, the best methods (that is, the most effective as giving tests of existence) are those of Boole, modified and developed by Imschenetsky, and that of Goursat, initially based upon the theory of characteristics, but subsequently brought into the form of Jacobian systems of simultaneous partial equations of the first order. These methods are exceptions to the foregoing general statement. But for equations of the second order or of higher orders, which involve two independent variables and in no case possess an intermediary integral, the most general methods are that of Ampère and that of Darboux, with such modifications and reconstruction as have been introduced by other writers; and though in these developments partial differential equations of the first order are introduced, still initially the subsidiary system is in effect a system with one independent variable expressed and the other, suppressed during the integration, playing a parametric part. In other words, the subsidiary system practically has one independent variable fewer than the original equation.

In another paper* I have given a method for dealing with partial differential equations of the second order in three variables when they possess an intermediary integral; and references will there be found to other writers upon the subject. My aim in the present paper has been to obtain a method for partial differential equations of the second order in three variables when, in general, they possess no intermediary integral. The natural generalisation of the idea in Darboux's method has been adopted, viz., the construction of subsidiary equations in which the number of expressed independent variables is less by unity than the number in the original equation; consequently the number is two. The subsidiary equations thus are a set of simultaneous partial differential equations in two independent variables and a number of dependent variables.

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* "Partial Differential Equations of the Second Order, involving Three Independent Variables and possessing an Intermediary Integral," Camb. Phil. Trans., vol. xvi., 1898, pp. 191–218.

It then appears that such differential equations of the second order having no intermediary integral divide themselves into two classes, discriminated by the distinction that, for the one class, what I have called the characteristic invariant, viz.:

$$p^2 \frac{\partial F}{\partial a} + pq \frac{\partial F}{\partial h} + q^2 \frac{\partial F}{\partial b} - p \frac{\partial F}{\partial g} - q \frac{\partial F}{\partial f} + \frac{\partial F}{\partial c} = 0,$$

can, *quâ* function of p and q, be resolved into two linear equations, and, for the other class, the characteristic invariant is irreducible.

The first section of this paper is devoted to the general theory of equations of the second order, so as to construct systems of equations subsidiary to the integration; occasional paragraphs in the other sections develop the general theory in connection with particular types of equations. The second section is devoted to the integration of equations whose characteristic invariant is reducible: a method is devised whereby the integration can be effected in those cases where the integral can be expressed in a finite form without partial quadratures; and various examples are given in elucidation of detailed processes. The third section is devoted to the integration of equations whose characteristic invariant is irreducible; and the method is applied with considerable detail to some of the equations that are important in mathematical physics.

It should be added that the case of three independent variables has been selected for detailed treatment, as being that of complexity next greater than the case of two independent variables, the general theory of which is fairly complete. An inspection of the results, as well as of the processes, will make it manifest that, for many of them, generalisation to the case of n independent variables is immediate.*

SECTION I.

General Theory.

1. Let the number of independent variables be three, and denote them by x, y, z. Denote the dependent variable by v, and write

$$\frac{\partial v}{\partial x} = l, \qquad \frac{\partial v}{\partial y} = m, \qquad \frac{\partial v}{\partial z} = n,$$

$$\frac{\partial^2 v}{\partial x^2} = a, \qquad \frac{\partial^2 v}{\partial y^2} = b, \qquad \frac{\partial^2 v}{\partial z^2} = c,$$

$$\frac{\partial^2 v}{\partial y\, \partial z} = f, \qquad \frac{\partial^2 v}{\partial z\, \partial x} = g, \qquad \frac{\partial^2 v}{\partial x\, \partial y} = h.$$

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* See a note at the end of the paper.

Let any general differential equation of the second order be taken in the form

$$F\,(v,\,x,\,y,\,z,\,l,\,m,\,n,\,a,\,b,\,c,\,f,\,g,\,h) = 0.$$

When the proper value of v is substituted, this becomes an identity, so that, when differentiated with regard to x, y, z, in succession, the results are identities. Hence, writing

$$\left.\begin{aligned} F_x &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial v}l + a\frac{\partial F}{\partial l} + h\frac{\partial F}{\partial m} + g\frac{\partial F}{\partial n} \\ F_y &= \frac{\partial F}{\partial y} + \frac{\partial F}{\partial v}m + h\frac{\partial F}{\partial l} + b\frac{\partial F}{\partial m} + f\frac{\partial F}{\partial n} \\ F_z &= \frac{\partial F}{\partial z} + \frac{\partial F}{\partial v}n + g\frac{\partial F}{\partial l} + f\frac{\partial F}{\partial m} + c\frac{\partial F}{\partial n} \end{aligned}\right\},$$

we have

$$\left.\begin{aligned} F_x + F_a\frac{\partial a}{\partial x} + F_b\frac{\partial b}{\partial x} + \ldots + F_h\frac{\partial h}{\partial x} &= 0 \\ F_y + F_a\frac{\partial a}{\partial y} + F_b\frac{\partial b}{\partial y} + \ldots + F_h\frac{\partial h}{\partial y} &= 0 \\ F_z + F_a\frac{\partial a}{\partial z} + F_b\frac{\partial b}{\partial z} + \ldots + F_h\frac{\partial h}{\partial z} &= 0 \end{aligned}\right\}.$$

By CAUCHY's theorem, a solution of $F = 0$ exists, determined by the values of v and one of its derivatives, assigned for a relation between x, y, z. This implies that, at all points on the surface represented by the relation, the values of v and, say, $\partial v/\partial x$ are given; and consequently the values of $\partial v/\partial y$ and $\partial v/\partial z$ are known at all points on the surface.

Taking now v, l, m, n, as known on the surface, and denoting by p, q, the derivatives of z with regard to x, y, along the surface, we have

$$\begin{aligned} dl &= adx + hdy + gdz = (a + pg)\,dx + (h + qg)\,dy, \\ dm &= hdx + bdy + fdz = (h + pf)\,dx + (b + qf)\,dy, \\ dn &= gdx + fdy + cdz = (g + pc)\,dx + (f + qc)\,dy, \end{aligned}$$

so that, as l, m, n, are known everywhere on the surface, the quantities

$$\begin{aligned} a + pg &= \frac{dl}{dx}, & h + qg &= \frac{dl}{dy}, \\ h + pf &= \frac{dm}{dx}, & b + qf &= \frac{dm}{dy}, \\ g + pc &= \frac{dn}{dx}, & f + qc &= \frac{dn}{dy}, \end{aligned}$$

are known along the surface. These equations require the relation

$$\frac{dl}{dy} + p\frac{dn}{dy} = \frac{dm}{dx} + q\frac{dn}{dx},$$

so that they determine five of the quantities a, b, c, f, g, h, in terms of the remaining one, say

$$\begin{aligned}
f &= \frac{dn}{dy} - qc, \\
b &= \frac{dm}{dy} - q\frac{dn}{dy} + q^2c, \\
g &= \frac{dn}{dx} - pc, \\
a &= \frac{dl}{dx} - p\frac{dn}{dx} + p^2c, \\
\left.\begin{aligned} h &= \frac{dm}{dx} - p\frac{dn}{dy} + pqc \\ &= \frac{dl}{dy} - q\frac{dn}{dx} + pqc \end{aligned}\right\}&.
\end{aligned}$$

We also have

$$F(v, x, y, z, l, m, n, a, b, c, f, g, h) = 0,$$

so that, in general, there are six equations to determine the six derivatives of the second order; and if F is algebraical in a, b, c, f, g, h, there will be a limited number of sets of values of these quantities, which can therefore be regarded as known along the surface.

In the same way, the derivatives of higher order can be deduced everywhere on the surface, and so, taking any point as an initial point, we have the values of all the derivatives of v known there; we then have a series in powers of $x - x_0$, $y - y_0$, $z - z_0$, which, in CAUCHY's theorem, is proved a converging series when $x_0y_0z_0$ is an ordinary point in space for the equation: and consequently we infer the existence of the solution as established by CAUCHY's theorem.*

This conclusion is justified only, however, if the equations do actually determine sets of values of a, b, c, f, g, h. In the case where sets of values are not determined, so that, *e.g.*, the equation $F = 0$ becomes evanescent on the substitution of the values of a, b, f, g, h, the preceding inference is not justified. Manifestly one such condition will be

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* The most general form of the theorem may be stated as follows:—

If $\phi(x, y, z) = 0$ be an ordinary relation for the equation $F = 0$, that is, if it is not a solution of the characteristic invariant equation, then a solution v of the equation $F = 0$ exists satisfying the conditions:—

(i) v is equal to a given arbitrary function of x, y, z, everywhere along the surface $\phi = 0$;

(ii) one of the derivatives of v is equal to a given arbitrary function of x, y, z, everywhere along the same surface;

and a solution satisfying these conditions is uniquely determined by them.

See also a paper, Proc. Lond. Math. Soc., vol. xxix, 1898, pp. 5–13, in particular, p. 12.

$$p^2\frac{\partial F}{\partial a}+pq\frac{\partial F}{\partial h}+q^2\frac{\partial F}{\partial b}-p\frac{\partial F}{\partial g}-q\frac{\partial F}{\partial f}+\frac{\partial F}{\partial c}=0\,;$$

but this is, of course, only one among a number of equations.

The method, practically contained in CAUCHY's theorem, leads to a result only in an infinite form; moreover, it takes no account of the alternative when conditions are not satisfied. We proceed, accordingly, to give another method, suggested by the corresponding investigation (due to DARBOUX) for equations of the second order in two variables.

2. The principle underlying DARBOUX's method is, in effect, similar to that which underlies AMPÈRE'S; but as DARBOUX's method most easily admits of application to equations in which the derivatives of highest order occur linearly, it is customary to form derivatives of a given equation, in order to secure that the equations discussed shall possess this property. If however the equation be already in a linear form, a mere generalisation of AMPÈRE'S method can first be tried, for the number of subsidiary equations is considerably smaller than in the other method, and consequently the integrations (if they can be performed) are correspondingly easier.* It will be sufficient for the present purpose to consider a particular example, say

$$b-f+g-h=0.$$

When we substitute for b, f, g, h, we find

$$\frac{dm}{dy}+(p-q-1)\frac{dn}{dy}+\frac{dn}{dx}-\frac{dm}{dx}+(q^2-pq+q-p)\,c=0\,;$$

the equation now must not determine the value of c, and so we must have

$$q^2-pq+q-p=0,$$

$$\frac{dm}{dy}+(p-q-1)\frac{dn}{dy}+\frac{dn}{dx}-\frac{dm}{dx}=0\,:$$

and there is also the identical condition

$$\frac{dm}{dx}-p\frac{dn}{dy}-\frac{dl}{dy}+q\frac{dn}{dx}=0.$$

From the first of these, it follows that either

$$p-q=0,$$

or

$$q+1=0.$$

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* A discussion of the subsidiary system, and of the relation of its integral to the solution of the original equation, will be found later, in §§ 27–29.

When $p - q = 0$, then z is a function of $x + y$. Also the second equation becomes

$$\frac{d}{dy}(m - n) = \frac{d}{dx}(m - n),$$

so that $m - n$ is a function of $x + y$. We infer, from the general considerations adduced, that we may take

$$m - n = \Phi(x + y, z),$$

where Φ is an arbitrary function.

When $q + 1 = 0$, then $y + z$ is a function of x. Also the second equation is

$$\frac{dm}{dy} - \frac{dm}{dx} + \frac{dn}{dx} + p\frac{dn}{dy} = 0,$$

which, by means of the identical condition, can be transformed to

$$-\frac{dl}{dy} + \frac{dm}{dy} = 0,$$

so that $l - m$ is a function of x, and the corresponding inference is that

$$l - m = \Theta(x, y + z),$$

where Θ is an arbitrary function.

Each of these is an intermediary integral, and the integration can be completed.*

3. Passing now to the generalisation of DARBOUX's method, we change the variables from x, y, z, to x, y, u, where u is a function of x, y, z, as yet undetermined, so that also z is a function of x, y, u, not yet determined. For the consequent variations of z when x, y, u, vary, write

$$\frac{dz}{dx} = p, \qquad \frac{dz}{dy} = q;$$

and when x, y, u, are the variables, denote the variations of the other quantities by d. Thus

$$\begin{array}{lll}
\frac{dv}{dx} = l + np, & \frac{dv}{dy} = m + nq, & \frac{dv}{du} = n\frac{dz}{du}, \\
\frac{dl}{dx} = a + gp, & \frac{dl}{dy} = h + gq, & \frac{dl}{du} = g\frac{dz}{du}, \\
\frac{dm}{dx} = h + fp, & \frac{dm}{dy} = b + fq, & \frac{dm}{du} = f\frac{dz}{du}, \\
\frac{dn}{dx} = g + cp, & \frac{dn}{dy} = f + cq, & \frac{dn}{du} = c\frac{dz}{du}.
\end{array}$$

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* The example is discussed (and the solution completed) in another connection in § 9.

Derivatives of the third order will be required. Write

$$\alpha_0 = \frac{\partial^3 v}{\partial x^3}, \qquad \beta_0 = \frac{\partial^3 v}{\partial x\,\partial y}, \qquad \gamma_0 = \frac{\partial^3 v}{\partial x\,\partial y^2}, \qquad \delta_0 = \frac{\partial^3 v}{\partial y^3},$$

$$\alpha_1 = \frac{\partial^3 v}{\partial x^2\,\partial z}, \qquad \beta_1 = \frac{\partial^3 v}{\partial x\,\partial y\,\partial z}, \qquad \gamma_1 = \frac{\partial^3 v}{\partial y^2\,\partial z},$$

$$\alpha_2 = \frac{\partial^3 v}{\partial x\,\partial z^2}, \qquad \beta_2 = \frac{\partial^3 v}{\partial y\,\partial z^2},$$

$$\alpha_3 = \frac{\partial^3 v}{\partial z^3},$$

so that

$$\left.\begin{aligned} da &= \alpha_0\,dx + \beta_0\,dy + \alpha_1\,dz \\ dh &= \beta_0\,dx + \gamma_0\,dy + \beta_1\,dz \\ dg &= \alpha_1\,dx + \beta_1\,dy + \alpha_2\,dz \\ db &= \gamma_0\,dx + \delta_0\,dy + \gamma_1\,dz \\ df &= \beta_1\,dx + \gamma_1\,dy + \beta_2\,dz \\ dc &= \alpha_2\,dx + \beta_2\,dy + \alpha_3\,dz \end{aligned}\right\}.$$

Now, from the equation $\mathrm{F} = 0$, we have, on supposing a proper value of v substituted, an identity, which thus admits of being differentiated and leading to identities. Thus

$$\frac{d\mathrm{F}}{dx} + \frac{\partial \mathrm{F}}{\partial a}\alpha_0 + \frac{\partial \mathrm{F}}{\partial h}\beta_0 + \frac{\partial \mathrm{F}}{\partial g}\alpha_1 + \frac{\partial \mathrm{F}}{\partial b}\gamma_0 + \frac{\partial \mathrm{F}}{\partial f}\beta_1 + \frac{\partial \mathrm{F}}{\partial c}\alpha_2 = 0,$$

or say

and also

$$\left.\begin{aligned} \mathrm{X} + \mathrm{A}\alpha_0 + \mathrm{H}\beta_0 + \mathrm{G}\alpha_1 + \mathrm{B}\gamma_0 + \mathrm{F}\beta_1 + \mathrm{C}\alpha_2 &= 0 \\ \mathrm{Y} + \mathrm{A}\beta_0 + \mathrm{H}\gamma_0 + \mathrm{G}\beta_1 + \mathrm{B}\delta_0 + \mathrm{F}\gamma_1 + \mathrm{C}\beta_2 &= 0 \\ \mathrm{Z} + \mathrm{A}\alpha_1 + \mathrm{H}\beta_1 + \mathrm{G}\alpha_2 + \mathrm{B}\gamma_1 + \mathrm{F}\beta_2 + \mathrm{C}\alpha_3 &= 0 \end{aligned}\right\}.$$

Consider the forms taken by these equations when the new variables are used. We have

$$\frac{da}{dx} = \alpha_0 + \alpha_1 p, \qquad \frac{da}{dy} = \beta_0 + \alpha_1 q, \qquad \frac{da}{du} = \alpha_1 \frac{dz}{du},$$

$$\frac{dh}{dx} = \beta_0 + \beta_1 p, \qquad \frac{dh}{dy} = \gamma_0 + \beta_1 q, \qquad \frac{dh}{du} = \beta_1 \frac{dz}{du},$$

$$\frac{dg}{dx} = \alpha_1 + \alpha_2 p, \qquad \frac{dg}{dy} = \beta_1 + \alpha_2 q, \qquad \frac{dg}{du} = \alpha_2 \frac{dz}{du},$$

$$\frac{db}{dx} = \gamma_0 + \gamma_1 p, \qquad \frac{db}{dy} = \delta_0 + \gamma_1 q, \qquad \frac{db}{du} = \gamma_1 \frac{dz}{du},$$

$$\frac{df}{dx} = \beta_1 + \beta_2 p, \qquad \frac{df}{dy} = \gamma_1 + \beta_2 q, \qquad \frac{df}{du} = \beta_2 \frac{dz}{du},$$

$$\frac{dc}{dx} = \alpha_2 + \alpha_3 p, \qquad \frac{dc}{dy} = \beta_2 + \alpha_3 q, \qquad \frac{dc}{du} = \alpha_3 \frac{dz}{du}.$$

The following relations, free from derivatives of the third order and not involving derivatives with regard to u, subsist among the derivatives of a, b, c, f, g, h, viz. :

$$\left.\begin{aligned} \frac{dh}{dx} - \frac{da}{dy} &= \beta_1 p - \alpha_1 q = p\frac{dg}{dy} - q\frac{dg}{dx} \\ \frac{db}{dx} - \frac{dh}{dy} &= \gamma_1 p - \beta_1 q = p\frac{df}{dy} - q\frac{df}{dx} \\ \frac{df}{dx} - \frac{dg}{dy} &= \beta_2 p - \alpha_2 q = p\frac{dc}{dy} - q\frac{dc}{dx} \end{aligned}\right\}.$$

Further, from those equations, we have

$$\begin{aligned} \alpha_0 &= -\beta_1\frac{p^2}{q} - \frac{p}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right) + \frac{da}{dx}, \\ \alpha_1 &= \beta_1\frac{p}{q} + \frac{1}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right), \\ \alpha_2 &= -\beta_1\frac{1}{q} + \frac{1}{q}\frac{dg}{dy}, \\ \alpha_3 &= \beta_1\frac{1}{pq} + \frac{1}{p}\frac{dc}{dx} - \frac{1}{pq}\frac{dg}{dy}, \\ \beta_0 &= -\beta_1 p + \frac{dh}{dx}, \\ \beta_2 &= -\beta_1\frac{1}{p} + \frac{1}{p}\frac{df}{dx}, \\ \gamma_0 &= -\beta_1 q + \frac{dh}{dy}, \\ \gamma_1 &= \beta_1\frac{q}{p} + \frac{1}{p}\left(\frac{db}{dx} - \frac{dh}{dy}\right), \\ \delta_0 &= -\beta_1\frac{q^2}{p} + \frac{db}{dy} - \frac{q}{p}\left(\frac{db}{dx} - \frac{dh}{dy}\right). \end{aligned}$$

Other expressions are obtainable, but they are equivalent to this set in virtue of the three relations above given. We take the value of β_1 to be

$$\frac{dh}{du} \div \frac{dz}{du}.$$

Substituting these values in the three equations, we find

$$\begin{aligned} -\beta_1\frac{\Delta}{q} + \mathrm{X} + \mathrm{A}\frac{da}{dx} - \mathrm{A}\frac{p}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right) + \mathrm{H}\frac{dh}{dx} + \mathrm{G}\frac{1}{q}\left(\frac{da}{dy} - \frac{dh}{dx}\right) \\ + \mathrm{B}\frac{dh}{dy} + \mathrm{C}\frac{1}{q}\frac{dg}{dy} = 0, \end{aligned}$$

$$-\beta_1\frac{\Delta}{p}+\mathrm{Y}+\mathrm{H}\frac{dh}{dy}+\mathrm{B}\frac{db}{dy}-\mathrm{B}\frac{q}{p}\left(\frac{db}{dx}-\frac{dh}{dy}\right)+\mathrm{F}\frac{1}{p}\left(\frac{db}{dx}-\frac{dh}{dy}\right)$$

$$+\mathrm{A}\frac{dh}{dx}+\mathrm{C}\frac{1}{p}\frac{df}{dx}=0,$$

$$\beta_1\frac{\Delta}{pq}+\mathrm{Z}+\mathrm{A}\frac{1}{q}\left(\frac{da}{dy}-\frac{dh}{dx}\right)+\mathrm{G}\frac{1}{q}\frac{dg}{dy}+\mathrm{B}\frac{1}{p}\left(\frac{db}{dx}-\frac{dh}{dy}\right)+\mathrm{F}\frac{1}{p}\frac{df}{dx}$$

$$+\mathrm{C}\frac{1}{p}\frac{dc}{dx}-\mathrm{C}\frac{1}{pq}\frac{dg}{dy}=0,$$

where

$$\Delta=\mathrm{A}p^2+\mathrm{H}pq+\mathrm{B}q^2-\mathrm{G}p-\mathrm{F}q+\mathrm{C};$$

and in each of these β_1 has the above-mentioned value.

4. There are two considerations, initially distinct, but found in the course of the argument to be concurrent, which enable us to obtain a certain set of subsidiary equations; they correspond to the two modes of obtaining the subsidiary equations in AMPÈRE's method of solving equations of the second order in two independent variables.

According to the first of them, we note that the new variable u is as yet limited by no conditions; it has hitherto remained arbitrary. Suppose it chosen so that $\Delta=0$, that is,

$$\mathrm{A}p^2+\mathrm{H}pq+\mathrm{B}q^2-\mathrm{G}p-\mathrm{F}q+\mathrm{C}=0.$$

Then the term in β_1 disappears from the three equations; and these (after some reductions in which $\Delta=0$ is used as well as the identical relations affecting the derivatives of a, b, c, f, g, h) take the forms

$$\left.\begin{aligned}
\xi&=\mathrm{X}+\mathrm{A}\left(\frac{da}{dx}-p\frac{dg}{dx}\right)+\mathrm{H}\left(\frac{dh}{dx}-p\frac{dg}{dy}\right)+\mathrm{B}\left(\frac{dh}{dy}-q\frac{dg}{dy}\right)+\mathrm{G}\frac{dg}{dx}+\mathrm{F}\frac{dg}{dy}=0\\
\eta&=\mathrm{Y}+\mathrm{A}\left(\frac{dh}{dx}-p\frac{df}{dx}\right)+\mathrm{H}\left(\frac{db}{dx}-p\frac{df}{dy}\right)+\mathrm{B}\left(\frac{db}{dy}-q\frac{df}{dy}\right)+\mathrm{G}\frac{df}{dx}+\mathrm{F}\frac{df}{dy}=0\\
\zeta&=\mathrm{Z}+\mathrm{A}\left(\frac{dg}{dx}-p\frac{dc}{dx}\right)+\mathrm{H}\left(\frac{df}{dx}-p\frac{dc}{dy}\right)+\mathrm{B}\left(\frac{df}{dy}-q\frac{dc}{dy}\right)+\mathrm{G}\frac{dc}{dx}+\mathrm{F}\frac{dc}{dy}=0
\end{aligned}\right\}.$$

According to the second of the considerations indicated, we assume that the new variable u, which has been adopted, is an argument in an arbitrary function that occurs in the solution. Then β_1 will, through the term dh/du in its value $dh/du \div dz/du$, introduce a triple differentiation with regard to u beyond any differentiation that occurs in the integral equations, while no one of the other terms in any of the equations will introduce more than a corresponding double differentiation with regard to u. Assuming the integral to be of such a form that these differentiations give rise to derivatives of the arbitrary function, it follows* that β_1 will contain a derivative with regard to u of the arbitrary function in question, of higher order than any other term in any of the equations. Now the equations must be satisfied identically when the value of v is substituted in them; hence the term in β_1 must disappear in and by itself in each case, that is, we have

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* Provided always that the number of derivatives of the arbitrary function in question, as occurring

$$\Delta = 0,$$

the same conclusion as before. The remaining parts of the equations must also vanish; their forms are already given.

5. The quantities, which have to be determined for the present purpose, are a, b, c, f, g, h, l, m, n, v, z, viz., eleven in all. They are functions of x, y, u. Omitting those equations in which derivatives with regard to u occur, the eleven quantities are to be functions of x and y. Constants that arise in the integration are constant because the variation of u does not appear explicitly; that is to say, the constants are functions of u.

The equations for the determination of the eleven unknowns are partial differential equations of the first order; their aggregate is constituted as follows:

First, for the equations defining quantities, we have

$$\frac{dv}{dx} = l + np, \qquad \frac{dv}{dy} = m + nq,$$
$$\frac{dl}{dx} = a + gp, \qquad \frac{dl}{dy} = h + gq,$$
$$\frac{dm}{dx} = h + fp, \qquad \frac{dm}{dy} = b + fq,$$
$$\frac{dn}{dx} = g + cp, \qquad \frac{dn}{dy} = f + cq,$$

eight in all.

But there are certain relations among derivatives that must be satisfied. We have

$$\frac{d}{dy}\left(\frac{dv}{dx}\right) = \frac{d}{dx}\left(\frac{dv}{dy}\right),$$

that is,

$$\frac{d}{dy}(l + np) = \frac{d}{dx}(m + nq),$$

or, since

$$\frac{dp}{dy} = \frac{d}{dy}\left(\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{dz}{dy}\right) = \frac{dq}{dx},$$

we have

$$\frac{dl}{dy} + p\frac{dn}{dy} = \frac{dm}{dx} + q\frac{dn}{dx},$$

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in the solution, is finite. If the solution is not expressible in finite terms, the inference is not necessarily justified in the present connection; we should then fall back upon the first of the two arguments. An example will be found in §§ 41–43.

a relation that is satisfied identically in virtue of the defining equations. As it is deduced from two of these equations, and it is satisfied identically in virtue of others, the inference is that the set of equations must consequently be reduced by one in number when only those which are independent are to be retained.

Treating the other three in the same manner, we find

$$\left.\begin{aligned}\frac{dh}{dx}+q\frac{dg}{dx}&=\frac{da}{dy}+p\frac{dg}{dy}\\ \frac{db}{dx}+q\frac{df}{dx}&=\frac{dh}{dy}+p\frac{df}{dy}\\ \frac{df}{dx}+q\frac{dc}{dx}&=\frac{dg}{dy}+p\frac{dc}{dy}\end{aligned}\right\},$$

equivalent to the identities obtained in § 3. As these are deduced from the eight defining equations, they are satisfied in virtue of those eight; they do not constitute any addition to the aggregate. Seven, therefore, is the number in this class.

The remainder are the equations characteristic of $F = 0$, viz., they are

$$\Delta = 0, \qquad \xi = 0, \qquad \eta = 0, \qquad \zeta = 0,$$

being four in all. It thus appears that the tale of independent partial differential equations in the system is eleven, being the same as the number of quantities to be determined.

It is to be noted—the verification is simple—that the original equation

$$F = 0$$

is an integral of these eleven simultaneous equations. Hence, for their effective solution, other ten integrals would be required if further considerations cannot be introduced; but it will appear from examples that this can be done, having the effect of appreciably shortening the process of integration.

6. One generalisation is immediately suggested by the results obtained. In solving equations of the second order in two independent variables, the subsidiary system is composed of a set of simultaneous equations involving one independent variable in effect; and the preceding investigation shows that, for equations of the second order in three independent variables, a subsidiary system can be constructed in the form of a set of simultaneous equations involving two independent variables in effect. It is thus suggested—and it is easy to see that the suggestion can be established definitely —that, *for an equation of the second order involving* n *independent variables, a subsidiary system can be constructed in the form of a set of simultaneous partial differential equations of the first order involving in effect* n − 1 *independent variables and a number of dependent variables,* this system being subsidiary to the integration of the proposed equation.

7. The first of the equations, $\Delta = 0$, belonging specially to the postulated equation $F = 0$, can be expressed in a different form. The quantity z is regarded as a function of x, y, u; and p, q, denote the values of dz/dx, dz/dy, respectively, when u is considered constant. Let the equation connecting z, x, y, u, be given (or taken) in the form

$$u = u(x, y, z);$$

then we have

$$\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z} = 0, \qquad \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z} = 0.$$

Substituting for p and q, the equation $\Delta = 0$ becomes

$$A\left(\frac{\partial u}{\partial x}\right)^2 + H\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + B\left(\frac{\partial u}{\partial y}\right)^2 + G\frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + F\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + C\left(\frac{\partial u}{\partial z}\right)^2 = 0,$$

which, after the preceding explanations in § 4, is an equation satisfied by an argument of an arbitrary function in the integral of the differential equation.

It is not difficult to prove that this equation is invariantive for all changes of the independent variables. For suppose them changed according to the transformations

$$\begin{aligned} x' &= \xi(x, y, z), \\ y' &= \eta(x, y, z), \\ z' &= \zeta(x, y, z); \end{aligned}$$

and let $\xi_x, \xi_y, \xi_z, \ldots$ denote derivatives of $\xi, \ldots$ while $l', m', n', a' \ldots$ denote derivatives of v with regard to the new variables. Then

$$l, m, n = \left(\begin{array}{ccc} \xi_x, & \eta_x, & \zeta_x \\ \xi_y, & \eta_y, & \zeta_y \\ \xi_z, & \eta_z, & \zeta_z \end{array}\right)\!\!\left(l', m', n'\right);$$

and

$$\begin{aligned} a &= (a', b', c', f', g', h')(\xi_x, \eta_x, \zeta_x)^2 + \ldots, \\ h &= (a', b', c', f', g', h')(\xi_x, \eta_x, \zeta_x)(\xi_y, \eta_y, \zeta_y) + \ldots, \\ &\ldots, \end{aligned}$$

the terms represented by $+ \ldots$ being terms involving the derivatives l', m', n', of the first order only. If, then, the differential equation

$$F(a, b, c, f, g, h, l, m, n, x, y, z) = 0$$

becomes

$$F'(a', b', \ldots \qquad\qquad) = 0$$

after these substitutions are made, we have

$$A' = A\xi_x^2 + H\xi_x\xi_y + B\xi_y^2 + G\xi_x\xi_y + F\xi_y\xi_z + C\xi_z^2,$$

$$H' = 2A\xi_x\eta_x + H(\xi_x\eta_y + \xi_y\eta_x) + 2B\xi_y\eta_y$$
$$+ G(\xi_x\eta_z + \xi_z\eta_x) + F(\xi_y\eta_z + \xi_z\eta_y) + 2C\xi_z\eta_z,$$

and so on; so that, if

$$u(x, y, z) = u'(x', y', z'),$$

we have

$$A'\left(\frac{\partial u'}{\partial \xi}\right)^2 + H'\frac{\partial u'}{\partial \xi}\frac{\partial u'}{\partial \eta} + \ldots = A\left(\frac{\partial u}{\partial x}\right)^2 + H\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \ldots$$

on substitution and collection of terms. The equation may therefore be called the *characteristic invariant* of the original differential equation.

8. A method for integrating partial differential equations of the second order, when they possess an intermediary integral, has been given by me elsewhere; its aim is the actual derivation of the intermediary integral which, being of the first order, can be regarded as soluble. The preceding method makes no assumption as to the existence of an intermediary integral, and indeed is entirely independent of that existence; so that it can be applied not merely to that former class, but also to equations that do not satisfy the preliminary conditions for the possession of an intermediary integral.

SECTION II.

Equations having a Resoluble Characteristic Invariant.

9. As a first example (which, it will be seen, possesses intermediary integrals), consider the equation

$$b = f - g + h.$$

The characteristic equation is

$$-pq + q^2 - p + q = 0,$$

which can be resolved into the two equations

$$p - q = 0, \quad q + 1 = 0.$$

The other three equations, deduced as in §§ 3, 4, are easily found to be

$$\left.\begin{aligned} -\frac{dh}{dx} + \frac{dh}{dy} + \frac{dg}{dx} + (p - q - 1)\frac{dg}{dy} &= 0 \\ -\frac{db}{dx} + \frac{db}{dy} + \frac{df}{dx} + (p - q - 1)\frac{df}{dy} &= 0 \\ -\frac{df}{dx} + \frac{df}{dy} + \frac{dc}{dx} + (p - q - 1)\frac{dc}{dy} &= 0 \end{aligned}\right\};$$

and we have the relations of identity, viz.,

$$\left.\begin{aligned} \frac{dh}{dx} - \frac{da}{dy} &= p\frac{dg}{dy} - q\frac{dg}{dx} \\ \frac{db}{dx} - \frac{dh}{dy} &= p\frac{df}{dy} - q\frac{df}{dx} \\ \frac{df}{dx} - \frac{dg}{dy} &= p\frac{dc}{dy} - q\frac{dc}{dx} \end{aligned}\right\} .$$

Take first the form

$$p - q = 0,$$

showing that z is any function of $x + y$. The three deduced equations then become

$$\begin{aligned} \frac{d}{dx}(g - h) &= \frac{d}{dy}(g - h), \\ \frac{d}{dx}(b - f) &= \frac{d}{dy}(b - f), \\ \frac{d}{dx}(c - f) &= \frac{d}{dy}(c - f), \end{aligned}$$

so that $h - g$, $b - f$, $c - f$, are functions of $x + y$.

Hence, as, by the original equation, we must have

$$h - g = b - f,$$

we take, as integrals of the differential equations of the present type,

$$\begin{aligned} h - g &= \mathrm{F}_1(x + y, z), \\ b - f &= \mathrm{F}_1(x + y, z), \\ c - f &= \mathrm{F}_2(x + y, z), \end{aligned}$$

where F_1 and F_2 are arbitrary. Hence also

$$\begin{aligned} \frac{\partial}{\partial x}(m - n) &= h - g = \quad \mathrm{F}_1(x + y, z), \\ \frac{\partial}{\partial y}(m - n) &= b - f = \quad \mathrm{F}_1(x + y, z), \\ \frac{\partial}{\partial z}(m - n) &= f - c = - \mathrm{F}_2(x + y, z). \end{aligned}$$

It therefore follows that

$$\frac{\partial \mathrm{F}_1}{\partial z} = - \frac{\partial \mathrm{F}_2}{\partial x} = - \frac{\partial \mathrm{F}_2}{\partial y};$$

and consequently there exists a function, say $\Phi(x + y, z)$, such that

$$F_1 = \frac{\partial \Phi}{\partial x}, \qquad F_1 = \frac{\partial \Phi}{\partial y}, \qquad F_2 = -\frac{\partial \Phi}{\partial z},$$

and therefore we have

$$m - n = \Phi(x + y, z),$$

where Φ is an arbitrary function. We might proceed from this equation to the primitive.

Next, take the relation

$$q + 1 = 0,$$

deduced from the characteristic; this shows that $z + y$ is a function of x. The three deduced equations now are

$$\left.\begin{aligned} -\frac{dh}{dx} + \frac{dh}{dy} + \frac{dg}{dx} + p\frac{dg}{dy} &= 0 \\ -\frac{db}{dx} + \frac{db}{dy} + \frac{df}{dx} + p\frac{df}{dy} &= 0 \\ -\frac{df}{dx} + \frac{df}{dy} + \frac{dc}{dx} + p\frac{dc}{dy} &= 0 \end{aligned}\right\};$$

and inserting the value $q = -1$ in the three relations of identity, they become

$$\left.\begin{aligned} \frac{dh}{dx} - \frac{da}{dy} &= p\frac{dg}{dy} + \frac{dg}{dx} \\ \frac{db}{dx} - \frac{dh}{dy} &= p\frac{df}{dy} + \frac{df}{dx} \\ \frac{df}{dx} - \frac{dg}{dy} &= p\frac{dc}{dy} + \frac{dc}{dx} \end{aligned}\right\}.$$

Combining these, we find

$$\frac{dh}{dy} - \frac{da}{dy} = 0, \qquad \frac{db}{dy} - \frac{dh}{dy} = 0, \qquad \frac{df}{dy} - \frac{dg}{dy} = 0,$$

so that $h - a$, $b - h$, $f - g$, are functions of x. Hence, as we must have

$$f - g = b - h$$

by the original equation we take for an integral equation, as in the former case,

$$\begin{aligned} f - g = b - h &= G_1(x, y + z), \\ a - h &= G_2(x, y + z), \end{aligned}$$

where G_1 and G_2 are arbitrary. But

$$\begin{aligned}\frac{\partial}{\partial x}(m-l) &= h-a = -G_2(x, y+z),\\ \frac{\partial}{\partial y}(m-l) &= b-h = \quad G_1(x, y+z),\\ \frac{\partial}{\partial z}(m-l) &= f-g = \quad G_1(x, y+z),\end{aligned}$$

so that we have

$$\frac{\partial G_1}{\partial x} = -\frac{\partial G_2}{\partial y} = -\frac{\partial G_2}{\partial z};$$

consequently there exists a function, say $-\Theta(x, y+z)$, such that

$$G_2 = +\frac{\partial\Theta}{\partial x}, \quad G_1 = -\frac{\partial\Theta}{\partial y} = -\frac{\partial\Theta}{\partial z},$$

and we then have

$$l - m = \Theta(x, y+z).$$

We may proceed from this equation to the primitive.

As two distinct intermediary integrals, viz.:

$$\begin{aligned}l - m &= \Theta(x, y+z),\\ m - n &= \Phi(x+y, z),\end{aligned}$$

have been obtained, it is worth noticing that they can be treated simultaneously, for they verify identically the Jacobi-Poisson condition of coexistence. If we introduce two new functions, θ and ϕ, defined by the equations

$$\begin{aligned}\Theta(\xi, \eta) &= \frac{\partial\theta}{\partial\xi} - \frac{\partial\theta}{\partial\eta},\\ \Phi(\xi, \eta) &= \frac{\partial\phi}{\partial\xi} - \frac{\partial\phi}{\partial\eta},\end{aligned}$$

so that θ and ϕ are arbitrary functions, then the simultaneous integral is easily obtained, say by Mayer's method,[*] in the form

$$v = \theta(x+y, z) + \phi(x, y+z),$$

which is the general primitive.

10. We next proceed to an example in which the given equation does not possess an intermediary integral. It is not difficult to construct differential equations

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[*] Math. Ann., vol. 5 (1872), pp. 460–466; see also my 'Theory of Differential Equations,' Part I., §§ 41–43.

of the second order in three independent variables, possessing a general primitive involving a couple of arbitrary functions of two arguments, but not possessing an intermediary integral.

Let

$$\frac{\partial}{\partial u}\phi(u, v) = \phi_1, \qquad \frac{\partial}{\partial v}\phi(u, v) = \phi_2,$$

$$\frac{\partial^2}{\partial u^2}\phi(u, v) = \phi_{11}, \qquad \frac{\partial^2}{\partial u\,\partial v}\phi(u, v) = \phi_{12}, \qquad \frac{\partial^2}{\partial v^2}\phi(u, v) = \phi_{22},$$

and so on for higher derivatives and for other functions. Take two functions

$$\phi = \phi(x + y, z), \qquad \theta = \theta(x + z, y);$$

and consider the equation

$$v = \phi + \rho\phi_1 + \sigma\phi_2 + \theta + \lambda\theta_1 + \mu\theta_2,$$

as one from which ϕ and θ are to be eliminated by means of derivatives of order not higher than two; the quantities ρ, σ, λ, μ, being defined as

$$\rho = a_1x + b_1y + c_1z, \qquad \lambda = \alpha x + \beta y + \gamma z,$$
$$\sigma = a'_1x + b'_1y + c'_1z, \qquad \mu = \alpha' x + \beta' y + \gamma' z.$$

We have

$$l = (1 + \alpha_1)\phi_1 + \alpha'_1\phi_2 + \rho\phi_{11} + \sigma\phi_{12} + (1 + \alpha)\theta_1 + \alpha'\theta_2 + \lambda\theta_{11} + \mu\theta_{12},$$
$$m = (1 + b_1)\phi_1 + b'_1\phi_2 + \rho\phi_{11} + \sigma\phi_{12} + \beta\theta_1 + (1 + \beta')\theta_2 + \lambda\theta_{12} + \mu\theta_{22},$$
$$n = c_1\phi_1 + (1 + c'_1)\phi_2 + \rho\phi_{12} + \sigma\phi_{22} + (1 + \gamma)\theta_1 + \gamma'\theta_2 + \lambda\theta_{11} + \mu\theta_{12}.$$

For second derivatives, it is unnecessary to form expressions for b and c, for the latter gives the only equation which contains ϕ_{222}, and the former the only one which contains θ_{222}, so that, when elimination is to be performed, these equations would be ignored. We therefore take

$$a = (1 + 2a_1)\phi_{11} + 2a'_1\phi_{12} + \rho\phi_{111} + \sigma\phi_{112} + (1 + 2\alpha)\theta_{11} + 2\alpha'\theta_{12} + \lambda\theta_{111} + \mu\theta_{112},$$

$$h = (1 + a_1 + b_1)\phi_{11} + (a'_1 + b_1)\phi_{12} + \rho\phi_{111} + \sigma\phi_{112}$$
$$+ \beta\theta_{11} + (1 + \alpha + \beta')\theta_{12} + \alpha'\theta_{22} + \lambda\theta_{112} + \mu\theta_{122},$$

$$g = c_1\phi_{11} + (1 + a_1 + c'_1)\phi_{12} + a'_1\phi_{22} + \rho\phi_{112} + \sigma\phi_{122}$$
$$+ (1 + \alpha + \gamma)\theta_{11} + (\alpha' + \gamma')\theta_{12} + \lambda\theta_{111} + \mu\theta_{112},$$

$$f = c_1\phi_{11} + (1 + b_1 + c'_1)\phi_{12} + b'_1\phi_{22} + \rho\phi_{112} + \sigma\phi_{122}$$
$$+ \beta\theta_{11} + (1 + \beta' + \gamma)\theta_{12} + \gamma'\theta_{22} + \lambda\theta_{112} + \mu\theta_{122}.$$

What is required is the elimination of the functional forms from these equations, if this be possible.

Manifestly all the third derivatives of ϕ and θ disappear in the combination $a+f-g-h$; in fact,

$$\begin{aligned} a+f-g-h = {} & (a_1-b_1)\phi_{11}-(a_1-b_1-a'_1+b'_1)\phi_{12}-(a'_1-b'_1)\phi_{22} \\ & +(\alpha-\gamma)\theta_{11}-(\alpha-\gamma-\alpha'+\gamma')\theta_{12}-(\alpha'-\gamma')\theta_{22}. \end{aligned}$$

If, by means of the expressions for l, m, n, it be possible to eliminate θ and ϕ, we must have a relation of the form

$$a+f-g-h+\xi l+\eta m+\zeta n=0,$$

where ξ, η, ζ, do not involve θ or ϕ.

In order that the terms in ϕ_1 and ϕ_2 may disappear, we have

$$(1+a_1)\xi+(1+b_1)\eta+c_1\zeta=0, \qquad a'_1\xi+b'_1\eta+(1+c'_1)\zeta=0;$$

that those in ϕ_{11}, ϕ_{12}, ϕ_{22}, may disappear, we have

$$\begin{gathered} a_1-b_1+\rho(\xi+\eta)=0, \qquad -(a'_1-b'_1)+\sigma\zeta=0, \\ -(a_1-b_1)+(a'_1-b'_1)+\sigma(\xi+\eta)+\rho\zeta=0; \end{gathered}$$

that those in θ_1 and θ_2 may disappear, we have

$$(1+\alpha)\xi+\beta\eta+(1+\gamma)\zeta=0, \qquad \alpha'\xi+(1+\beta')\eta+\gamma'\zeta=0;$$

and, finally, that those in θ_{11}, θ_{12}, θ_{22}, may disappear, we have

$$\begin{gathered} \alpha-\gamma+\lambda(\xi+\zeta)=0, \qquad -(\alpha'-\gamma')+\mu\eta=0, \\ -(\alpha-\gamma)+(\alpha'-\gamma')+\mu(\xi+\zeta)+\lambda\eta=0. \end{gathered}$$

These equations are to be satisfied simultaneously.

Using the last set of three, we have, on substituting in the third from the first and second,

$$(\lambda+\mu)(\xi+\eta+\zeta)=0;$$

and similarly, from the first set of three,

$$(\rho+\sigma)(\xi+\eta+\zeta)=0.$$

We accordingly take

$$\xi+\eta+\zeta=0.$$

With this value, the first set gives

$$\frac{a_1 - b_1}{\rho} - \frac{a'_1 - b'_1}{\sigma} = 0,$$

that is,

$$(a'_1 - b'_1)(a_1 x + b_1 y + c_1 z) = (a_1 - b_1)(a'_1 x + b'_1 y + c'_1 z).$$

Since x, y, z, are independent variables, this can be satisfied only if

$$\frac{a'_1}{a_1} = \frac{b'_1}{b_1} = \frac{c'_1}{c_1} = k, \text{ say},$$

so that

$$\sigma = k\rho.$$

Similarly, the second set gives

$$\frac{\alpha - \gamma}{\lambda} - \frac{\alpha' - \gamma'}{\mu} = 0,$$

leading to

$$\frac{\alpha'}{\alpha} = \frac{\beta'}{\beta} = \frac{\gamma'}{\gamma} = \kappa, \text{ say},$$

so that

$$\mu = \kappa\lambda.$$

The first set of equations can now be replaced by

$$\begin{aligned} \xi + \eta + \zeta &= 0, \\ a_1 - b_1 - \rho\zeta &= 0, \\ \xi + \eta + a_1\xi + b_1\eta + c_1\zeta &= 0, \\ \zeta + k(a_1\xi + b_1\eta + c_1\zeta) &= 0. \end{aligned}$$

Hence $k = -1$, and so $\sigma = -\rho$; also

$$a_1\xi + b_1\eta + (c_1 - 1)\zeta = 0,$$

so that

$$\frac{\xi}{b_1 - c_1 + 1} = \frac{\eta}{c_1 - a_1 - 1} = \frac{\zeta}{a_1 - b_1} = \frac{1}{a_1 x + b_1 y + c_1 z}.$$

Similarly, by the other set, we find $\kappa = -1$, and so $\mu = -\lambda$; and

$$\frac{\xi}{1 - \beta + \gamma} = \frac{\eta}{\alpha - \gamma} = \frac{\zeta}{-\alpha + \beta - 1} = \frac{1}{\alpha x + \beta y + \gamma z}.$$

As the values of ξ, η, ζ, must be the same in the two determinations, and as the variables x, y, z, are independent, we have

$$\frac{\alpha}{a_1} = \frac{\beta}{b_1} = \frac{\gamma}{c_1} = \frac{1 - \beta + \gamma}{1 + b_1 - c_1} = \frac{\alpha - \gamma}{c_1 - a_1 - 1} = \frac{\beta - \alpha - 1}{a_1 - b_1} = \frac{1}{p}, \text{ say,}$$

whence

$$\rho = p\lambda, \qquad \beta = \alpha + \tfrac{1}{2}, \qquad \gamma = \alpha + \frac{1}{2p},$$

so that

$$\lambda = \alpha (x + y + z) + \tfrac{1}{2} y + \frac{1}{2p} z.$$

It is easy to verify that, when p is distinct from 0 and ∞, there is no intermediary integral, that is, no relation between l, m, n, involving only one of the arbitrary functions θ and ϕ. We have

$$\frac{\xi}{1 + \dfrac{1}{p}} = \frac{\eta}{-\dfrac{1}{p}} = \frac{\zeta}{-1} = \frac{1}{2\lambda};$$

and then the differential equation is

$$a + f - g - h + \frac{1}{2\lambda} \left\{ \left(1 + \frac{1}{p}\right) l - \frac{1}{p} m - n \right\} = 0.$$

It has no intermediary integral. Its primitive is

$$v = \phi + \theta + \lambda \{ p (\phi_1 - \phi_2) + \theta_1 - \theta_2 \},$$

where

$$\phi = \phi (x + y, z), \qquad \theta = \theta (x + z, y);$$

and λ has the above value, and p is neither 0 nor ∞.

11. Now take a particular case, so as to illustrate the method of integration. Let

$$\alpha = 0, \qquad p = 1;$$

then

$$\lambda = \tfrac{1}{2} (y + z).$$

The differential equation is

$$a + f - g - h + \frac{2l - m - n}{y + z} = 0;$$

and it is required to obtain the primitive

$$v = \phi + \theta + \tfrac{1}{2} (y + z) (\phi_1 - \phi_2 + \theta_1 - \theta_2),$$

where

$$\phi = \phi (x + y, z), \qquad \theta = \theta (x + z, y).$$

For the differential equation thus postulated, the characteristic equation is

$$p^2 - pq + p - q = 0,$$

that is,

$$(p - q)(p + 1) = 0.$$

We therefore have two solutions. The first is

$$p - q = 0,$$

leading to

$$z = \text{function of } x + y;$$

the second is

$$p + 1 = 0,$$

leading to

$$y = \text{function of } x + z.$$

The other three equations, particular to the equation under consideration, are

$$\begin{aligned}
&\frac{da}{dx} - p\frac{dg}{dx} - \frac{dh}{dx} + p\frac{dg}{dy} - \frac{dg}{dx} + \frac{dg}{dy} + \frac{2a - h - g}{y + z} &&= 0,\\
&\frac{dh}{dx} - p\frac{df}{dx} - \frac{db}{dx} + p\frac{df}{dy} - \frac{df}{dx} + \frac{df}{dy} + \frac{2h - b - f}{y + z} - \frac{2l - m - n}{(y + z)^2} &&= 0,\\
&\frac{dg}{dx} - p\frac{dc}{dx} - \frac{df}{dx} + p\frac{dc}{dy} - \frac{dc}{dx} + \frac{dc}{dy} + \frac{2g - f - c}{y + z} - \frac{2l - m - n}{(y + z)^2} &&= 0,
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{d}{dx}(a - h) - (p + 1)\left(\frac{dg}{dx} - \frac{dg}{dy}\right) + \frac{2a - h - g}{y + z} &&= 0,\\
&\frac{d}{dx}(h - b) - (p + 1)\left(\frac{df}{dx} - \frac{df}{dy}\right) + \frac{2h - b - f}{y + z} - \frac{2l - m - n}{(y + z)^2} &&= 0,\\
&\frac{d}{dx}(g - f) - (p + 1)\left(\frac{dc}{dx} - \frac{dc}{dy}\right) + \frac{2g - f - c}{y + z} - \frac{2l - m - n}{(y + z)^2} &&= 0.
\end{aligned}$$

Taking first the case

$$p + 1 = 0,$$

we have

$$\frac{d}{dx}(a - h - g + f) + \frac{2a - h - g - 2g + f + c}{y + z} + \frac{2l - m - n}{(y + z)^2} = 0.$$

But

$$2a - h - g - 2g + f + c = \left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial z}\right)(2l - m - n) = \frac{d}{dx}(2l - m - n),$$

and

$$\frac{d}{dx}\frac{1}{y + z} = -\frac{1}{(y + z)^2}p = \frac{1}{(y + z)^2};$$

so that

$$\frac{d}{dx}(a - h - g + f) + \frac{d}{dx}\left(\frac{2l - m - n}{y + z}\right) = 0.$$

We thus recover the differential equation, which is an integral of the system ; the arbitrary function, which would arise through the integration, is definite: we have, in fact,

$$a - h - g + f + \frac{2l - m - n}{y + z} = 0.$$

Using this integral, the second equation becomes (on the elimination of $2l - m - n$)

$$\frac{d}{dx}(h - b) + \frac{a + h - b - g}{y + z} = 0.$$

Combining this with the first equation, we have

$$\frac{d}{dx}(a - 2h + b) + \frac{a - 2h + b}{y + z} = 0,$$

so that, as $p + 1 = 0$, we have

$$\frac{d}{dx}\left(\frac{a - 2h + b}{y + z}\right) = 0.$$

Since $p + 1 = 0$ implies that $z + x$ is a function of y, we infer that

$$\frac{a - 2h + b}{y + z} = \text{arb. fn. of } y,$$

when

$$z + x = \text{arb. fn. of } y\,;$$

and consequently an integral that can be associated with the original equation is given by

$$\frac{a - 2h + b}{y + z} = \theta(z + x, y),$$

where θ is an arbitrary function.

Taking next the case $p = q$, the alternative that arises out of the characteristic equation, we have the three other equations the same as before. It is now necessary to take account of the three equations of identity, which, when the relation $p = q$ is used, are of the form

$$\frac{dh}{dx} - \frac{da}{dy} = p\left(\frac{dg}{dy} - \frac{dg}{dx}\right),$$

$$\frac{db}{dx} - \frac{dh}{dy} = p\left(\frac{df}{dy} - \frac{df}{dx}\right),$$

$$\frac{df}{dx} - \frac{dg}{dy} = p\left(\frac{dc}{dy} - \frac{dc}{dx}\right);$$

so that, eliminating the terms in p from the three equations, we have

$$\frac{d}{dx}(a-g) - \frac{d}{dy}(a-g) + \frac{2a-h-g}{y+z} = 0,$$

$$\frac{d}{dx}(h-f) - \frac{d}{dy}(h-f) + \frac{2h-b-f}{y+z} - \frac{2l-m-n}{(y+z)^2} = 0,$$

$$\frac{d}{dx}(g-c) - \frac{d}{dy}(g-c) + \frac{2g-f-c}{y+z} - \frac{2l-m-n}{(y+z)^2} = 0.$$

Eliminating the term in $2l-m-n$ from the second and the third of these by means of the original differential equation, we obtain the modified equations in the forms

$$\frac{d}{dx}(h-f) - \frac{d}{dy}(h-f) + \frac{a+h-b-g}{y+z} = 0,$$

$$\frac{d}{dx}(g-c) - \frac{d}{dy}(g-c) + \frac{a-h+g-c}{y+z} = 0.$$

By the first of the former and the second of the latter, we find

$$\frac{d}{dx}(a-2g+c) - \frac{d}{dy}(a-2g+c) + \frac{a-2g+c}{y+z} = 0.$$

Now, as $p = q$, z behaves like a constant under the operation $d/dx - d/dy$; hence we have

$$\left(\frac{d}{dx} - \frac{d}{dy}\right)\left\{\frac{a-2g+c}{y+z}\right\} = 0,$$

when

$$\left(\frac{d}{dx} - \frac{d}{dy}\right) z = 0.$$

Consequently

$$\frac{a-2g+c}{y+z} = \text{arb. fn. of } x+y,$$

when

$$z = \text{arb. fn. of } x+y;$$

and we therefore infer that

$$\frac{a-2g+c}{y+z} = \phi(x+y, z)$$

is an integral that can be associated with the original differential equation, ϕ being an arbitrary function.

In order to proceed to the primitive, we take first

$$a - 2g + c = (y+z)\,\phi(x+y, z),$$

and introduce a new arbitrary function f, defined by

$$\phi = f_{111} - 3f_{112} + 3f_{122} - f_{222},$$

so that

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial z}\right)^2 v=(y+z)\{f_{111}-3f_{112}+3f_{122}-f_{222}\},$$

where f is an arbitrary function of $x+y$, z. Hence

$$l-n=\frac{\partial v}{\partial x}-\frac{\partial v}{\partial z}=(y+z)(f_{11}-2f_{12}+f_{22})+f_1-f_2+\mathrm{G}(x+z,\,y),$$

where G is arbitrary so far as this equation is concerned.

Similarly, introducing a new arbitrary function g, defined by

$$\theta(x+z,\,y)=\theta=g_{111}-3g_{112}+3g_{122}-g_{222},$$

the equation

$$a-2h+b=(y+z)\,\theta(x+z,\,y)$$

leads to the equation

$$a-2h+b=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^2 v=(y+z)(g_{111}-3g_{112}+3g_{122}-g_{222}).$$

Hence

$$l-m=\frac{\partial v}{\partial x}-\frac{\partial v}{\partial y}=(y+z)(g_{11}-2g_{12}+g_{22})+g_1-g_2+\mathrm{F}(x+y,\,z),$$

where F is arbitrary so far as this equation is concerned.

In order that the two equations, giving the values of $l-n$ and $l-m$ respectively, may coexist, they must satisfy the POISSON-JACOBI condition (U, V) = 0, which, when developed, gives

$$f_1-f_2+\mathrm{G}+g_1-g_2+\mathrm{F}=0\,;$$

so that, taking account of the arbitrary character of the functions, we have

$$\mathrm{F}=-(f_1-f_2),\qquad \mathrm{G}=-(g_1-g_2).$$

Thus

$$\begin{aligned} l-n&=(y+z)(f_{11}-2f_{12}+f_{22})+f_1-f_2-(g_1-g_2),\\ l-m&=(y+z)(g_{11}-2g_{12}+g_{22})-(f_1-f_2)+g_1-g_2. \end{aligned}$$

It is easy to verify, not merely that these equations coexist, but also that each of them satisfies the differential equation; but neither is an intermediary integral in the customary sense, for each of them includes two arbitrary functions of two arguments.

The equations are of the first order; it is easy to obtain the primitive in the form

$$v=2f+2g+(y+z)(f_1-f_2+g_1-g_2),$$

where $f_1=f(x+y,\,z)$ and $g=g(x+z,\,y)$ are arbitrary functions, and f_1, f_2, g_1, g_2, are their respective first derivatives.

12. It will be noticed that in these examples the equation $\Delta = 0$ (which is of the second degree in p and q) is resoluble, so that it can be replaced by two linear equations, and that the latter have, in turn, been combined with the other equations of the system. Now, these equations are of LAGRANGE'S linear form, and their integral is such that some combination θ of variables can be an arbitrary function of some other combination ϕ. Further, it has appeared that the integral of the subsidiary system (other than the original equation) is such as to make some combination ψ of the variable quantities a functional combination of θ or ϕ at the same time that θ and ϕ are functionally related, so that, as the functional forms are arbitrary, we infer that

$$\psi = \Psi(\theta, \phi),$$

where Ψ is arbitrary, is an equation that can coexist with the original equation. Hence it is to be inferred that *when* $\Delta = 0$ *is a resoluble equation, that is, can be resolved into two equations linear in* p *and* q, *arbitrary functions of two arguments occur in the most general integral equivalent of the original equation.*

13. The converse also is true, viz., *if an integral relation involve at least one arbitrary function of a couple of distinct arguments and be equivalent to a partial differential equation of the second order, and not to an equation of order lower than the second freed from arbitrary functional forms, then the characteristic invariant equation can be resolved into two linear equations.* (The number of independent variables is, of course, presumed to be three.)

Let ξ and η be two independent functions of x, y, z, so that not more than one of the three quantities

$$\xi_x\eta_y - \eta_x\xi_y, \qquad \xi_y\eta_z - \xi_z\eta_y, \qquad \xi_z\eta_x - \eta_z\xi_x,$$

can vanish. As regards the arbitrary function of ξ and η, let it occur in the integral equation in the form

$$v = \Theta\{\ldots\ldots, \phi(\xi, \eta), \ldots\},$$

where ϕ denotes the derivative of the arbitrary function of highest order occurring in Θ. Then we have

$$\left.\begin{aligned} l &= \frac{\partial\Theta}{\partial\phi}(\phi_1\xi_x + \phi_2\eta_x) + \ldots \\ m &= \frac{\partial\Theta}{\partial\phi}(\phi_1\xi_y + \phi_2\eta_y) + \ldots \\ n &= \frac{\partial\Theta}{\partial\phi}(\phi_1\xi_z + \phi_2\eta_z) + \ldots \end{aligned}\right\},$$

$$a = \frac{\partial\Theta}{\partial\phi}\{\phi_{11}\xi_x^2 + 2\phi_{12}\xi_x\eta_x + \phi_{22}\eta_x^2\} + \text{derivatives of } \phi \text{ of lower order},$$

$$h = \frac{\partial\Theta}{\partial\phi}\{\phi_{11}\xi_x\xi_y + \phi_{12}(\xi_x\eta_y + \xi_y\eta_x) + \phi_{22}\eta_x\eta_y\} + \ldots,$$

and so for the others. Now the integral equation is, by hypothesis, equivalent to a partial differential equation of the second order, say to

$$F(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0\,;$$

hence when these values are substituted the equation is to be satisfied, and accordingly the terms involving the various combinations of the arbitrary functions must disappear. Thus the highest power of ϕ_{11}—or what in effect is the same thing, the highest power of $\phi_{11}\frac{\partial\Theta}{\partial\phi}$—must disappear of itself, and therefore

$$\xi_x^2\frac{\partial F}{\partial a} + \xi_y^2\frac{\partial F}{\partial b} + \xi_z^2\frac{\partial F}{\partial c} + \xi_x\xi_y\frac{\partial F}{\partial h} + \xi_y\xi_z\frac{\partial F}{\partial f} + \xi_z\xi_x\frac{\partial F}{\partial g} = 0,$$

or, with the former notation,

$$A\xi_x^2 + H\xi_x\xi_y + G\xi_x\xi_z + B\xi_y^2 + F\xi_y\xi_z + C\xi_z^2 = 0.$$

But the term involving the highest power of $\phi_{12}\frac{\partial\Theta}{\partial\phi}$, which will be of the same degree as the highest power of $\phi_{11}\frac{\partial\Theta}{\partial\phi}$, must also disappear of itself; and this gives rise to the equation

$$2A\xi_x\eta_x + H(\xi_x\eta_y + \xi_y\eta_x) + G(\xi_x\eta_z + \xi_z\eta_x) + 2B\xi_y\eta_y + F(\xi_y\eta_z + \xi_z\eta_y) + 2C\xi_z\eta_z = 0\,;$$

and likewise the term involving the highest power of $\phi_{22}\frac{\partial\Theta}{\partial\phi}$ must disappear, leading to the equation

$$A\eta_x^2 + H\eta_x\eta_y + G\eta_x\eta_z + B\eta_y^2 + F\eta_y\eta_z + C\eta_z^2 = 0.$$

From these we have

$$(A\xi_x + \tfrac{1}{2}H\xi_y + \tfrac{1}{2}G\xi_z)^2 = (\tfrac{1}{4}H^2 - AB)\,\xi_y^2 + 2\,(\tfrac{1}{4}GH - \tfrac{1}{2}AF)\,\xi_y\xi_z + (\tfrac{1}{4}G^2 - AC)\,\xi_z^2,$$

$$(A\eta_x + \tfrac{1}{2}H\eta_y + \tfrac{1}{2}G\eta_z)^2 = (\tfrac{1}{4}H^2 - AB)\,\eta_y^2 + 2\,(\tfrac{1}{4}GH - \tfrac{1}{2}AF)\,\eta_y\eta_z + (\tfrac{1}{4}G^2 - AC)\,\eta_z^2,$$

$$\begin{aligned}(A\xi_x + \tfrac{1}{2}H\xi_y + \tfrac{1}{2}G\xi_z)\,(A\eta_x + \tfrac{1}{2}H\eta_y + \tfrac{1}{2}G\eta_z)&\\ = (\tfrac{1}{4}H^2 - AB)\,\xi_y\eta_y + (\tfrac{1}{4}GH - \tfrac{1}{2}AF)\,(\xi_y\eta_z + \xi_z\eta_y) + (\tfrac{1}{4}G^2 - AC)\,\xi_z\eta_z\,;&\end{aligned}$$

so that, squaring the last, subtracting the product of the first two, reducing, and removing the factor A, we have

$$(\text{Discrt. of }\Delta)\,(\xi_y\eta_z - \xi_z\eta_y)^2 = 0.$$

Similarly, by taking modifications of the first equation in the form

$$(\tfrac{1}{2}H\xi_x + B\xi_y + \tfrac{1}{2}F\xi_z)^2 = \ldots,$$

or in the form

$$(\tfrac{1}{2}G\xi_x + \tfrac{1}{2}F\xi_y + C\xi_z)^2 = \ldots,$$

and removing factors B and C respectively, we find

$$(\text{Discrt. of } \Delta)\ (\xi_z\eta_x - \xi_x\eta_z)^2 = 0,$$
$$(\text{Discrt. of } \Delta)\ (\xi_x\eta_y - \xi_y\eta_x)^2 = 0.$$

Now it has been seen that not more than one of the three quantities

$$\xi_y\eta_z - \xi_z\eta_y, \qquad \xi_z\eta_x - \xi_x\eta_z, \qquad \xi_x\eta_y - \xi_y\eta_x,$$

can vanish. Consequently

$$\text{the Discriminant of } \Delta = 0,$$

in other words, the equation $\Delta = 0$, that is

$$\mathrm{A}p^2 + \mathrm{H}pq + \mathrm{B}q^2 - \mathrm{G}q - \mathrm{F}p + \mathrm{C} = 0,$$

can be resolved into two equations linear in p and q. This establishes the proposition.

But if ϕ, instead of being a function of two arguments ξ and η, were a function of only a single argument u, then instead of three equations we infer only one equation of the type

$$\mathrm{A}u_x{}^2 + \mathrm{H}u_xu_y + \mathrm{B}u_y{}^2 + \mathrm{G}u_zu_x + \mathrm{F}u_yu_z + \mathrm{C}u_z{}^2 = 0,$$

and its resolubility cannot be established. (It is, of course, not the case that it is not resoluble in particular cases; it is not resoluble in general.) Hence *when the equation* $\Delta = 0$ *cannot be resolved into two equations, linear in* p *and* q, *we infer that the arbitrary functions which occur in the integral equivalent are functions of only a single argument.*

14. In the case when $\Delta = 0$ is resoluble into two linear equations, and when the other equations possess integrable combinations, a method can be constructed for obtaining those combinations. Thus take the example considered in § 11, where the deduction of the combinations is fortuitous in the sense that no indication of the kind of combination is given. Let

$$\theta(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0$$

be an integrable combination; that is, we must have

$$\frac{d\theta}{dx} = 0, \qquad \frac{d\theta}{dy} = 0.$$

Either (i) one of them, or (ii) a linear cross between them, or (iii) both of them, must be satisfied in virtue of the set of equations.

Let us consider, first, the case when

$$p + 1 = 0$$

so that q remains arbitrary. Evidently

$$\frac{d\theta}{dy} = 0$$

will not be the equation to be satisfied; so that the effective combination must arise in the form

$$\frac{d\theta}{dx} = 0$$

when $p + 1 = 0$. In other words, we must have the equation

$$\frac{\partial\theta}{\partial x} + a\frac{\partial\theta}{\partial l} + h\frac{\partial\theta}{\partial m} + g\frac{\partial\theta}{\partial n} - \left(\frac{\partial\theta}{\partial z} + g\frac{\partial\theta}{\partial l} + f\frac{\partial\theta}{\partial m} + c\frac{\partial\theta}{\partial n}\right) + \frac{\partial\theta}{\partial v}(l - n)$$
$$+ \frac{\partial\theta}{\partial a}\frac{da}{dx} + \frac{\partial\theta}{\partial b}\frac{db}{dx} + \frac{\partial\theta}{\partial c}\frac{dc}{dx} + \frac{\partial\theta}{\partial f}\frac{df}{dx} + \frac{\partial\theta}{\partial g}\frac{dg}{dx} + \frac{\partial\theta}{\partial h}\frac{dh}{dx} = 0,$$

(where the value $p = -1$ has been inserted), satisfied in virtue of the system of subsidiary equations. The relations of identity all involve q, and therefore cannot be useful for the purpose. Hence the above equation must be a linear combination of

$$\left.\begin{aligned} \frac{da}{dx} - \frac{dh}{dx} + \mathrm{X} &= 0 \\ \frac{dh}{dx} - \frac{db}{dx} + \mathrm{Y} &= 0 \\ \frac{dg}{dx} - \frac{df}{dx} + \mathrm{Z} &= 0 \end{aligned}\right\},$$

where

$$\mathrm{X} = \frac{2a - h - g}{y + z},$$
$$\mathrm{Y} = \frac{2h - b - f}{y + z} - \frac{2l - m - n}{(y + z)^2},$$
$$\mathrm{Z} = \frac{2g - f - c}{y + z} - \frac{2l - m - n}{(y + z)^2};$$

these being the subsidiary equations particular to the present case when the value $p = -1$ is inserted.

When therefore we substitute

$$\frac{da}{dx} = \frac{dh}{dx} - \mathrm{X}, \qquad \frac{db}{dx} = \frac{dh}{dx} + \mathrm{Y}, \qquad \frac{df}{dx} = \frac{dg}{dx} + \mathrm{Z},$$

in the equation derived from θ, the latter should become an identity; that is, we should have

$$\frac{\partial\theta}{\partial x} + a\frac{\partial\theta}{\partial l} + h\frac{\partial\theta}{\partial m} + g\frac{\partial\theta}{\partial n} - \left(\frac{\partial\theta}{\partial z} + g\frac{\partial\theta}{\partial l} + f\frac{\partial\theta}{\partial m} + c\frac{\partial\theta}{\partial n}\right) + \frac{\partial\theta}{\partial a}\left(\frac{dh}{dx} - \mathrm{X}\right)$$
$$+ \frac{\partial\theta}{\partial b}\left(\frac{dh}{dx} + \mathrm{Y}\right) + \frac{\partial\theta}{\partial c}\frac{dc}{dx} + (l-n)\frac{\partial\theta}{\partial v} + \frac{\partial\theta}{\partial f}\left(\frac{dg}{dx} + \mathrm{Z}\right) + \frac{\partial\theta}{\partial g}\frac{dg}{dx} + \frac{\partial\theta}{\partial h}\frac{dh}{dx} = 0$$

satisfied independently of the values of derivatives of h, c, g, with regard to x. It therefore follows that, if a function θ of the suggested type should exist, it must satisfy the system of equations

$$\frac{\partial\theta}{\partial c} = 0,$$
$$\frac{\partial\theta}{\partial f} + \frac{\partial\theta}{\partial g} = 0,$$
$$\frac{\partial\theta}{\partial a} + \frac{\partial\theta}{\partial h} + \frac{\partial\theta}{\partial b} = 0,$$

and

$$\frac{\partial\theta}{\partial x} - \frac{\partial\theta}{\partial z} + (a-g)\frac{\partial\theta}{\partial l} + (h-f)\frac{\partial\theta}{\partial m} + (g-c)\frac{\partial\theta}{\partial n} + \frac{\partial\theta}{\partial v}(l-n)$$
$$- \mathrm{X}\frac{\partial\theta}{\partial a} + \mathrm{Y}\frac{\partial\theta}{\partial b} + \mathrm{Z}\frac{\partial\theta}{\partial f} = 0;$$

and the number of functionally independent solutions of these homogeneous simultaneous equations is the number of integrable combinations of the subsidiary system.

This system of partial differential equations of the first order must be rendered complete by associating with it the Jacobi-Poisson conditions. This complete system, obtained by the regular processes, is without difficulty proved to be equivalent to

$$\frac{\partial\theta}{\partial v} = 0, \qquad \frac{\partial\theta}{\partial c} = 0,$$
$$-\frac{\partial\theta}{\partial h} = \frac{\partial\theta}{\partial a} + \frac{\partial\theta}{\partial b},$$
$$\tfrac{1}{2}\frac{\partial\theta}{\partial l} = -\frac{\partial\theta}{\partial m} = -\frac{\partial\theta}{\partial n} = \frac{1}{y+z}\frac{\partial\theta}{\partial f} = -\frac{1}{y+z}\frac{\partial\theta}{\partial g} = \left(\frac{\partial\theta}{\partial a} - \frac{\partial\theta}{\partial b}\right)\frac{1}{y+z},$$

and

$$\frac{\partial\theta}{\partial x} - \frac{\partial\theta}{\partial z} - \frac{2l-m-n}{(y+z)^2}\frac{\partial\theta}{\partial a} - \frac{2a-3h+b+f-g}{y+z}\frac{\partial\theta}{\partial b} = 0,$$

the latter being the modification of the last of the four initial equations.

The complete system thus contains nine equations: it involves thirteen variables, viz., a, b, c, f, g, h, l, m, n, v, x, y, z; and consequently it possesses four functionally independent solutions. These can be obtained, by any of the regular methods, in the form

$$y,\ x + z,\ \frac{2a - 3h + b + f - g}{y + z} + \frac{2l - m - n}{(y + z)^2},$$
$$a - h - g + f + \frac{2l - m - n}{y + z}.$$

But the last is zero, owing to the original differential equation; and by using this imposed restriction, the second becomes

$$\frac{a - 2h + b}{y + z}.$$

Consequently the most general solution of the system is

$$\Phi\left(\frac{a - 2h + b}{y + z},\ x + z,\ y\right) = 0,$$

where Φ is arbitrary; an equivalent of this is

$$\frac{a - 2h + b}{y + z} = \theta\,(z + x,\ y),$$

where θ is arbitrary.

Similarly a new relation between derivatives of the second order can be deduced by taking the alternative solution $p - q = 0$ of the characteristic equation.

The rest of the solution proceeds as before when once the system of partial differential equations satisfied by

$$\vartheta = \vartheta\,(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0$$

is obtained. Now, when $p = q$, the relations of identity are

$$\frac{dh}{dx} - \frac{da}{dy} = p\left(\frac{dg}{dy} - \frac{dg}{dx}\right),$$
$$\frac{db}{dx} - \frac{dh}{dy} = p\left(\frac{df}{dy} - \frac{df}{dx}\right),$$
$$\frac{df}{dx} - \frac{dg}{dy} = p\left(\frac{dc}{dy} - \frac{dc}{dx}\right).$$

These can be used to eliminate p from the equations particular to the present case, and the latter then become

$$\left.\begin{aligned}
\frac{da}{dx} - \frac{da}{dy} - \left(\frac{dg}{dx} - \frac{dg}{dy}\right) + \mathrm{X} = 0 \\
\frac{dh}{dx} - \frac{dh}{dy} - \left(\frac{df}{dx} - \frac{df}{dy}\right) + \mathrm{Y} = 0 \\
\frac{dg}{dx} - \frac{dg}{dy} - \left(\frac{dc}{dx} - \frac{dc}{dy}\right) + \mathrm{Z} = 0
\end{aligned}\right\},$$

which hold whatever be the value of p. But from $\vartheta = 0$ we have

$$\frac{\partial\vartheta}{\partial x} + \frac{\partial\vartheta}{\partial v}(l + np) + \frac{\partial\vartheta}{\partial l}(a + gp) + \frac{\partial\vartheta}{\partial m}(h + fp) + \frac{\partial\vartheta}{\partial n}(g + cp)$$
$$+ \frac{\partial\vartheta}{\partial a}\frac{da}{dx} + \ldots + \frac{\partial\vartheta}{\partial c}\frac{dc}{dx} = 0,$$

and

$$\frac{\partial\vartheta}{\partial y} + \frac{\partial\vartheta}{\partial v}(m + nq) + \frac{\partial\vartheta}{\partial l}(h + gq) + \frac{\partial\vartheta}{\partial m}(b + fq) + \frac{\partial\vartheta}{\partial n}(f + cq)$$
$$+ \frac{\partial\vartheta}{\partial a}\frac{da}{dy} + \ldots + \frac{\partial\vartheta}{\partial c}\frac{dc}{dy} = 0.$$

For our immediate purpose, $p = q$; hence, subtracting, we have

$$\frac{\partial\vartheta}{\partial x} - \frac{\partial\vartheta}{\partial y} + \frac{\partial\vartheta}{\partial v}(l - m) + \frac{\partial\vartheta}{\partial l}(a - h) + \frac{\partial\vartheta}{\partial m}(h - b) + \frac{\partial\vartheta}{\partial n}(g - f)$$
$$+ \Sigma\frac{\partial\vartheta}{\partial a}\left(\frac{da}{dx} - \frac{da}{dy}\right) = 0,$$

which, being free from p and q, must be satisfied in virtue of the above three equations. This being the case, it must, when we substitute for

$$\frac{da}{dx} - \frac{da}{dy}, \qquad \frac{dh}{dx} - \frac{dh}{dy}, \qquad \frac{dc}{dx} - \frac{dc}{dy},$$

be satisfied independently of the values of

$$\frac{dh}{dx} - \frac{dh}{dy}, \qquad \frac{df}{dx} - \frac{df}{dy}, \qquad \frac{dg}{dx} - \frac{dg}{dy}.$$

Assigning the necessary conditions, we find

$$\frac{\partial\vartheta}{\partial b} = 0,$$

$$\frac{\partial\vartheta}{\partial f} + \frac{\partial\vartheta}{\partial h} = 0,$$

$$\frac{\partial\vartheta}{\partial a} + \frac{\partial\vartheta}{\partial g} + \frac{\partial\vartheta}{\partial c} = 0,$$

$$\frac{\partial\vartheta}{\partial x} - \frac{\partial\vartheta}{\partial y} + \frac{\partial\vartheta}{\partial v}(l - m) + \frac{\partial\vartheta}{\partial l}(a - h) + \frac{\partial\vartheta}{\partial m}(h - p) + \frac{\partial\vartheta}{\partial n}(g - f)$$
$$- \mathrm{X}\frac{\partial\vartheta}{\partial a} - \mathrm{Y}\frac{\partial\vartheta}{\partial h} + \mathrm{Z}\frac{\partial\vartheta}{\partial c} = 0.$$

The integration of these equations can, as already stated, be effected in the same way as for the preceding part of the solution of the original equation; the most general solution of the system is found to be

$$\frac{a - 2g + c}{y + z} = \phi\,(x + y,\, z).$$

15. That the method just expounded is not restricted to individual instances of equations, for which $\Delta = 0$ is resoluble, can be seen as follows.

We consider, more generally, the case when

$$\mathrm{A}p^2 + \mathrm{H}pq + \mathrm{B}q^2 - \mathrm{G}p - \mathrm{F}q + \mathrm{C} = 0$$

is resoluble into two linear equations. We have

$$(\mathrm{A}p + \tfrac{1}{2}\mathrm{H}q - \tfrac{1}{2}\mathrm{G})^2 = (\tfrac{1}{4}\mathrm{H}^2 - \mathrm{AB})\,q^2 + (\mathrm{AF} - \tfrac{1}{2}\mathrm{GH})\,q + \tfrac{1}{4}\mathrm{G}^2 - \mathrm{AC}.$$

Let

$$\tfrac{1}{4}\mathrm{H}^2 - \mathrm{AB} = \theta^2,$$

$$\mathrm{AF} - \tfrac{1}{2}\mathrm{GH} = -\,2\theta^2\phi\,;$$

then, since

$$(\tfrac{1}{4}\mathrm{H}^2 - \mathrm{AB})\,(\tfrac{1}{4}\mathrm{G}^2 - \mathrm{AC}) = \tfrac{1}{4}\,(\mathrm{AF} - \tfrac{1}{2}\mathrm{GH})^2,$$

we have

$$\tfrac{1}{4}\mathrm{G}^2 - \mathrm{AC} = \theta^2\phi^2,$$

and the equation is

$$\mathrm{A}p + \tfrac{1}{2}\mathrm{H}q - \tfrac{1}{2}\mathrm{G} = \pm\,\theta\,(q - \phi),$$

that is, we have the two equations

$$\left.\begin{aligned} \mathrm{A}p + (\tfrac{1}{2}\mathrm{H} - \theta)\,q - (\tfrac{1}{2}\mathrm{G} - \theta\phi) = 0 \\ \mathrm{A}p + (\tfrac{1}{2}\mathrm{H} + \theta)\,q - (\tfrac{1}{2}\mathrm{G} + \theta\phi) = 0 \end{aligned}\right\},$$

where

$$\left.\begin{aligned} \phi &= \frac{\tfrac{1}{4}\mathrm{GH} - \tfrac{1}{2}\mathrm{AF}}{\tfrac{1}{4}\mathrm{H}^2 - \mathrm{AB}} \\ \theta^2 &= \tfrac{1}{4}\mathrm{H}^2 - \mathrm{AB} \end{aligned}\right\}.$$

Taking the former of the two equations, viz.,

$$\mathrm{A}p + (\tfrac{1}{2}\mathrm{H} - \theta)\,q - (\tfrac{1}{2}\mathrm{G} - \theta\phi) = 0,$$

we seek to obtain combinations of the three equations of identity with the three equations particular to the present case. The first equations of each of these sets, as given in §§ 4, 5, are

$$\mathrm{X} + \mathrm{A}\frac{da}{dx} + \mathrm{H}\frac{dh}{dx} + \mathrm{B}\frac{dh}{dy} + \mathrm{G}\frac{dg}{dx} + \mathrm{F}\frac{dg}{dy} - p\left(\mathrm{A}\frac{dg}{dx} + \mathrm{H}\frac{dg}{dy}\right) - q\mathrm{B}\frac{dg}{dy} = 0,$$

$$\frac{da}{dy} - \frac{dh}{dx} + p\,\frac{dg}{dy} - q\,\frac{dg}{dx} = 0.$$

Multiply the latter by $\frac{1}{2}\mathrm{H} - \theta$, and add it to the former. In the resulting equation, the coefficient of dg/dx is

$$\mathrm{G} - \mathrm{A}p - (\tfrac{1}{2}\mathrm{H} - \theta)\, q,$$

which is

$$= (\tfrac{1}{2}\mathrm{G} + \theta\phi)\,;$$

the coefficient of dg/dy is

$$\begin{aligned} - \mathrm{H}p - \mathrm{B}q + (\tfrac{1}{2}\mathrm{H} - \theta)\, p + \mathrm{F} &= - \{(\tfrac{1}{2}\mathrm{H} + \theta)\, p + \mathrm{B}q\} + \mathrm{F} \\ &= - \frac{\frac{1}{2}\mathrm{H} + \theta}{\mathrm{A}} \{\mathrm{A}p + (\tfrac{1}{2}\mathrm{H} - \theta)\, q\} + \mathrm{F} \\ &= - \frac{\frac{1}{2}\mathrm{H} + \theta}{\mathrm{A}} (\tfrac{1}{2}\mathrm{G} - \theta\phi) + \mathrm{F}. \end{aligned}$$

But this coefficient, multiplied by A,

$$\begin{aligned} &= \mathrm{AF} - \tfrac{1}{4}\mathrm{GH} - \tfrac{1}{2}\mathrm{G}\theta + \tfrac{1}{2}\mathrm{H}\theta\phi + \theta^2\phi \\ &= \tfrac{1}{4}\mathrm{GH} - \tfrac{1}{2}\mathrm{G}\theta + \tfrac{1}{2}\mathrm{H}\theta\phi - \theta^2\phi \\ &= (\tfrac{1}{2}\mathrm{G} + \theta\phi)\,(\tfrac{1}{2}\mathrm{H} - \theta)\,; \end{aligned}$$

and therefore the terms, involving derivatives of g, are

$$(\tfrac{1}{2}\mathrm{G} + \theta\phi)\left(\frac{dg}{dx} + \frac{\frac{1}{2}\mathrm{H} - \theta}{\mathrm{A}}\,\frac{dg}{dy}\right)$$

The terms involving derivatives of a are

$$\mathrm{A}\left(\frac{da}{dx} + \frac{\frac{1}{2}\mathrm{H} - \theta}{\mathrm{A}}\,\frac{da}{dy}\right);$$

and those involving derivatives of h are

$$(\tfrac{1}{2}\mathrm{H} + \theta)\,\frac{dh}{dx} + \mathrm{B}\,\frac{dh}{dy} = (\tfrac{1}{2}\mathrm{H} + \theta)\left(\frac{dh}{dx} + \frac{\frac{1}{2}\mathrm{H} - \theta}{\mathrm{A}}\,\frac{dh}{dy}\right).$$

The equation is now in its simplest form. The other pairs may be treated in the same way; and thus, corresponding to the equation

$$\mathrm{A}p + (\tfrac{1}{2}\mathrm{H} - \theta)\, q - (\tfrac{1}{2}\mathrm{G} - \theta\phi) = 0,$$

we have a system of three subsidiary equations free from p and q in the form

$$\left.\begin{aligned} \mathrm{X} + \mathrm{A}\delta a + (\tfrac{1}{2}\mathrm{H} + \theta)\,\delta h + (\tfrac{1}{2}\mathrm{G} + \theta\phi)\,\delta g &= 0 \\ \mathrm{Y} + \mathrm{A}\delta h + (\tfrac{1}{2}\mathrm{H} + \theta)\,\delta b + (\tfrac{1}{2}\mathrm{G} + \theta\phi)\,\delta f &= 0 \\ \mathrm{Z} + \mathrm{A}\delta g + (\tfrac{1}{2}\mathrm{H} + \theta)\,\delta f + (\tfrac{1}{2}\mathrm{G} + \theta\phi)\,\delta c &= 0, \end{aligned}\right\},$$

where

$$\delta = \frac{d}{dx} + \frac{\frac{1}{2}\mathrm{H} - \theta}{\mathrm{A}}\,\frac{d}{dy}\,.$$

If

$$u(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0$$

be an integrable combination of these equations, then we must have

$$\begin{aligned}\frac{\partial u}{\partial a}\frac{da}{dx} + \ldots + \frac{\partial u}{\partial h}\frac{dh}{dx} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\\ + \frac{\partial u}{\partial v}(l + np) + \frac{\partial u}{\partial l}(a + gp) + \frac{\partial u}{\partial m}(h + fp) + \frac{\partial u}{\partial n}(g + cp) = 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial a}\frac{da}{dy} + \ldots + \frac{\partial u}{\partial h}\frac{dh}{dy} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q + \frac{\partial u}{\partial v}(m + nq) + \frac{\partial u}{\partial l}(h + gq)\\ + \frac{\partial u}{\partial m}(b + fq) + \frac{\partial u}{\partial n}(f + cq) = 0.\end{aligned}$$

Multiply the latter by $\frac{\frac{1}{2}\text{H} - \theta}{\text{A}}$, and add to the former; where p and q occur in the result, it is in the form

$$p + \frac{\frac{1}{2}\text{H} - \theta}{\text{A}}q = \frac{\frac{1}{2}\text{G} - \theta\phi}{\text{A}},$$

and so the equation is

$$\begin{aligned}\frac{\partial u}{\partial a}\delta a + \frac{\partial u}{\partial b}\delta b + \frac{\partial u}{\partial c}\delta c + \frac{\partial u}{\partial f}\delta f + \frac{\partial u}{\partial g}\delta g + \frac{\partial u}{\partial h}\delta h + \frac{\partial u}{\partial x} + \frac{\frac{1}{2}\text{H} - \theta}{\text{A}}\frac{\partial u}{\partial y} + \frac{\frac{1}{2}\text{G} - \theta\phi}{\text{A}}\frac{\partial u}{\partial z}\\ + \frac{\partial u}{\partial v}\left(l + \frac{\frac{1}{2}\text{H} - \theta}{\text{A}}m + \frac{\frac{1}{2}\text{G} - \theta\phi}{\text{A}}n\right) + \frac{\partial u}{\partial l}\left(a + \frac{\frac{1}{2}\text{H} - \theta}{\text{A}}h + \frac{\frac{1}{2}\text{G} - \theta\phi}{\text{A}}g\right)\\ + \frac{\partial u}{\partial m}\left(h + \frac{\frac{1}{2}\text{H} - \theta}{\text{A}}b + \frac{\frac{1}{2}\text{G} - \theta\phi}{\text{A}}f\right) + \frac{\partial u}{\partial n}\left(g + \frac{\frac{1}{2}\text{H} - \theta}{\text{A}}f + \frac{\frac{1}{2}\text{G} - \theta\phi}{\text{A}}c\right) = 0.\end{aligned}$$

This must be satisfied in virtue of the three preceding equations and independently of the actual values of $\delta a, \ldots, \delta f$. Substitute for δa, δb, δc; then the coefficients of δf, δg, δh, and the term independent of these must vanish. We thus obtain four linear homogeneous partial differential equations, viz.,

$$\left.\begin{aligned}0 &= \frac{\partial u}{\partial h} - \frac{\frac{1}{2}\text{H} + \theta}{\text{A}}\frac{\partial u}{\partial a} - \frac{\text{A}}{\frac{1}{2}\text{H} + \theta}\frac{\partial u}{\partial b}\\ 0 &= \frac{\partial u}{\partial g} - \frac{\frac{1}{2}\text{G} + \theta\phi}{\text{A}}\frac{\partial u}{\partial a} - \frac{\text{A}}{\frac{1}{2}\text{G} + \theta\phi}\frac{\partial u}{\partial c}\\ 0 &= \frac{\partial u}{\partial f} - \frac{\frac{1}{2}\text{G} + \theta\phi}{\frac{1}{2}\text{H} + \theta}\frac{\partial u}{\partial b} - \frac{\frac{1}{2}\text{H} + \theta}{\frac{1}{2}\text{G} + \theta\phi}\frac{\partial u}{\partial c}\\ 0 &= -\frac{\text{X}}{\text{A}} - \frac{\text{Y}}{\frac{1}{2}\text{H} + \theta} - \frac{\text{Z}}{\frac{1}{2}\text{G} + \theta\phi} + \frac{\partial u}{\partial x} + l\frac{\partial u}{\partial v} + a\frac{\partial u}{\partial l} + h\frac{\partial u}{\partial m} + g\frac{\partial u}{\partial n}\\ &\quad + \frac{\frac{1}{2}\text{H} - \theta}{\text{A}}\left(\frac{\partial u}{\partial y} + m\frac{\partial u}{\partial v} + h\frac{\partial u}{\partial l} + b\frac{\partial u}{\partial m} + f\frac{\partial u}{\partial n}\right)\\ &\quad + \frac{\frac{1}{2}\text{G} - \theta\phi}{\text{A}}\left(\frac{\partial u}{\partial z} + n\frac{\partial u}{\partial v} + g\frac{\partial u}{\partial l} + f\frac{\partial u}{\partial m} + c\frac{\partial u}{\partial n}\right)\end{aligned}\right\},$$

with

$$0 = \mathrm{A}p + (\tfrac{1}{2}\mathrm{H} - \theta)\, q - (\tfrac{1}{2}\mathrm{G} - \theta\phi).$$

This system of four equations must be rendered complete by constructing the additional equations that arise out of the JACOBI-POISSON conditions. If, when complete, the system contains n equations, then it possesses $13 - n$ functionally independent solutions. Among these must be included (i) the original differential equation

$$\mathrm{F}\,(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0\,;$$

(ii) the two distinct integrals of

$$\frac{dx}{\mathrm{A}} = \frac{dy}{\frac{1}{2}\mathrm{H} - \theta} = \frac{dz}{\frac{1}{2}\mathrm{G} - \theta\phi}\,;$$

say these are ξ, η.

Putting these on one side, there are thus $10 - n$ new functionally independent solutions. A not uncommon case is $n = 9$, when there is one new solution, say u. Then we have

$$u = \psi\,(\xi, \eta),$$

where ψ is an arbitrary functional form ; and this equation coexists with the original equation

$$\mathrm{F} = 0.$$

16. Thus far we have considered only one of the two equations into $\Delta = 0$ is resoluble. When we consider the other equation, viz.,

$$\mathrm{A}p + (\tfrac{1}{2}\mathrm{H} + \theta)\, q - (\tfrac{1}{2}\mathrm{Q} + \theta\phi) = 0,$$

the sole difference in the general analysis is manifestly a change in the sign of θ ; and we therefore obtain the corresponding system of linear homogeneous partial differential equations, determining an integral combination (if any), by changing the sign of θ in the preceding system. The method of integration is the same as before.

It may happen that neither of these two systems possesses a solution distinct from the differential equation. If, however, either (or both) should possess such a solution, then n must be less than 10, and certain conditions—viz., those in order that the system when complete should contain not more than nine equations—must be satisfied. These are the conditions in order that one equation—or two equations, if the result hold for both systems—of the second order involving an arbitrary function of two arguments should be associable with the given equation.

And it should be noted that the characteristic invariant of an equation associable with the given equation is satisfied by that linear equation in the characteristic invariant of the given equation which is used to derive the new equation. The result is general, and the proof of the general result is immediate.

17. Now it may happen that the simultaneous system of equations admits of no new common solution in either case; the inference then is that no equation of the second order containing a single arbitrary function can be associated with, or is compatible with, the given differential equation. But it may then be that some new equation of the third order—new, that is, in the sense that it is not one of the immediate derivatives of the given equation—containing an arbitrary function can be associated with the given equation; and this may occur with each of the linear factors of $\Delta = 0$. And so on, precisely as in DARBOUX's method for dealing with partial differential equations in two independent variables; we seek to obtain one equation (or, it may be, two equations) of finite order which are compatible with the given equation, contain one arbitrary function, and are not mere derivatives from that given equation.

We have been proceeding on the supposition that the equation possesses no intermediary integral. If no other equation of finite order is compatible with the given equation,* then the method ceases to be effective. In that case, the only result generally attainable seems at present to be that which occurs in the establishment of CAUCHY's existence-theorem; the integral certainly contains two arbitrary functions, but its expression (in the form of a converging series) is not finite.

18. Suppose that the conditions for the existence of a new common solution are satisfied for neither of the systems in §§ 15, 16, so that no new equation of the second order, containing only a single arbitrary function, is compatible with the given equation. We proceed to construct the system of subsidiary equations which determine an equation (if any) of the third order containing only one arbitrary function, and compatible with the given equation

$$\mathrm{F}\,(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0.$$

On account of this equation, we have three derived equations of the third order, viz., with the former notation

$$\left.\begin{aligned} \mathrm{X} + \mathrm{A}\alpha_0 + \mathrm{H}\beta_0 + \mathrm{G}\alpha_1 + \mathrm{B}\gamma_0 + \mathrm{F}\beta_1 + \mathrm{C}\alpha_2 = 0 \\ \mathrm{Y} + \mathrm{A}\beta_0 + \mathrm{H}\gamma_0 + \mathrm{G}\beta_1 + \mathrm{B}\delta_0 + \mathrm{F}\gamma_1 + \mathrm{C}\beta_2 = 0 \\ \mathrm{Z} + \mathrm{A}\alpha_1 + \mathrm{H}\beta_1 + \mathrm{G}\alpha_2 + \mathrm{B}\gamma_1 + \mathrm{F}\beta_2 + \mathrm{C}\alpha_3 = 0 \end{aligned}\right\};$$

and the new equation (if any) must be compatible with these.

19. The process is an amplification of that used in § 2. When the proper value of v is substituted in $\mathrm{F} = 0$, the latter becomes an identity, so that, when it is

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* A simple instance is given by

$$a - h - g + f + \lambda\frac{2l - m - n}{y + z} = 0,$$

where λ is a positive constant other than an integer.

differentiated with regard to the independent variables, the results are identities. By hypothesis, no new equation is derivable when first derivatives are formed: we therefore form the derivatives of the second order, being six in all; viz., they are

$$\frac{d^2\mathrm{F}}{dx^2} = 0, \quad \frac{d^2\mathrm{F}}{dy^2} = 0, \quad \frac{d^2\mathrm{F}}{dz^2} = 0,$$

$$\frac{d^2\mathrm{F}}{dy\,dz} = 0, \quad \frac{d^2\mathrm{F}}{dz\,dx} = 0, \quad \frac{d^2\mathrm{F}}{dx\,dy} = 0,$$

equations which contain derivatives of v of order 4. Let these fifteen derivatives be denoted by $s_1, s_2, \ldots, s_{15}$, their definitions being given by the scheme

	$dx +$	$dy +$	dz
$d\alpha_0 =$	r_1	r_2	r_3
$d\alpha_1$	r_3	r_5	r_6
$d\alpha_2$	r_6	r_9	r_{10}
$d\alpha_3$	r_{10}	r_{14}	r_{15}
$d\beta_0$	r_2	r_4	r_5
$d\beta_1$	r_5	r_8	r_9
$d\beta_2$	r_9	r_{13}	r_{14}
$d\gamma_0$	r_4	r_7	r_8
$d\gamma_1$	r_8	r_{12}	r_{13}
$d\delta_0$	r_7	r_{11}	r_{12}

Further, let (XX) denote the part of $\frac{d^2\mathrm{F}}{dx^2}$ which is free from derivatives of the fourth order, (XY) the corresponding part of $\frac{d^2\mathrm{F}}{dx\,dy}$, (XZ) that of $\frac{d^2\mathrm{F}}{dx\,dz}$, and so on. Then the six equations are

$$\left.\begin{aligned}
(\mathrm{XX}) + \mathrm{A}r_1 + \mathrm{H}r_2 + \mathrm{G}r_3 + \mathrm{B}r_4 + \mathrm{F}r_5 + \mathrm{C}r_6 &= 0 \\
(\mathrm{XY}) + \mathrm{A}r_2 + \mathrm{H}r_4 + \mathrm{G}r_5 + \mathrm{B}r_7 + \mathrm{F}r_8 + \mathrm{C}r_9 &= 0 \\
(\mathrm{XZ}) + \mathrm{A}r_3 + \mathrm{H}r_5 + \mathrm{G}r_6 + \mathrm{B}r_8 + \mathrm{F}r_9 + \mathrm{C}r_{10} &= 0 \\
(\mathrm{YY}) + \mathrm{A}r_4 + \mathrm{H}r_7 + \mathrm{G}r_8 + \mathrm{B}r_{11} + \mathrm{F}r_{12} + \mathrm{C}r_{13} &= 0 \\
(\mathrm{YZ}) + \mathrm{A}r_5 + \mathrm{H}r_8 + \mathrm{G}r_9 + \mathrm{B}r_{12} + \mathrm{F}r_{13} + \mathrm{C}r_{14} &= 0 \\
(\mathrm{ZZ}) + \mathrm{A}r_6 + \mathrm{H}r_9 + \mathrm{G}r_{10} + \mathrm{B}r_{13} + \mathrm{F}r_{14} + \mathrm{C}r_{15} &= 0
\end{aligned}\right\}.$$

As before, let the variables be changed from x, y, z, to x, y, u, where u is a function of x, y, z, as yet undetermined, whence also z is a function of x, y, u. For the consequent variations of z when x, y, u, vary, we write p and q for dz/dx, dz/dy, respectively; and we adopt the notation of § 3. The new expressions for the variations of $v, l, m, n, a, b, c, f, g, h$, are given in § 3; those for the variations of $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \delta_0$, are given by the equations

$$\left.\begin{array}{lll}
\dfrac{d\alpha_0}{dx} = r_1 + r_3 p, & \dfrac{d\alpha_0}{dy} = r_2 + r_3 q, & \dfrac{d\alpha_0}{du} = r_3 \dfrac{dz}{du} \\
\dfrac{d\alpha_1}{dx} = r_3 + r_6 p, & \dfrac{d\alpha_1}{dy} = r_5 + r_6 q, & \dfrac{d\alpha_1}{du} = r_6 \dfrac{dz}{du} \\
\dfrac{d\alpha_2}{dx} = r_6 + r_{10} p, & \dfrac{d\alpha_2}{dy} = r_9 + r_{10} q, & \dfrac{d\alpha_2}{du} = r_{10} \dfrac{dz}{du} \\
\dfrac{d\alpha_3}{dx} = r_{10} + r_{15} p, & \dfrac{d\alpha_3}{dy} = r_{14} + r_{15} q, & \dfrac{d\alpha_3}{du} = r_{15} \dfrac{dz}{du} \\
\dfrac{d\beta_0}{dx} = r_2 + r_5 p, & \dfrac{d\beta_0}{dy} = r_4 + r_5 q, & \dfrac{d\beta_0}{du} = r_5 \dfrac{dz}{du} \\
\dfrac{d\beta_1}{dx} = r_5 + r_9 p, & \dfrac{d\beta_1}{dy} = r_8 + r_9 q, & \dfrac{d\beta_1}{du} = r_9 \dfrac{dz}{du} \\
\dfrac{d\beta_2}{dx} = r_9 + r_{14} p, & \dfrac{d\beta_2}{dy} = r_{13} + r_{14} q, & \dfrac{d\beta_2}{du} = r_{14} \dfrac{dz}{du} \\
\dfrac{d\gamma_0}{dx} = r_4 + r_8 p, & \dfrac{d\gamma_0}{dy} = r_7 + r_8 q, & \dfrac{d\gamma_0}{du} = r_8 \dfrac{dz}{du} \\
\dfrac{d\gamma_1}{dx} = r_8 + r_{13} p, & \dfrac{d\gamma_1}{dy} = r_{12} + r_{13} q, & \dfrac{d\gamma_1}{dy} = r_{13} \dfrac{dz}{du} \\
\dfrac{d\delta_0}{dx} = r_7 + r_{12} p, & \dfrac{d\delta_0}{dy} = r_{11} + r_{12} q, & \dfrac{d\delta_0}{dy} = r_{12} \dfrac{dz}{du}
\end{array}\right\} .$$

The following relations subsist among the derivatives of $\alpha_0, \ldots, \delta_0$, free from derivatives of order 4 and from derivatives with regard to u, viz.,

$$\left.\begin{array}{l}
\dfrac{d\beta_0}{dx} - \dfrac{d\alpha_0}{dy} = p\dfrac{d\alpha_1}{dy} - q\dfrac{d\alpha_1}{dx} = r_5 p - r_3 q \\
\dfrac{d\beta_1}{dx} - \dfrac{d\alpha_1}{dy} = p\dfrac{d\alpha_2}{dy} - q\dfrac{d\alpha_2}{dx} = r_9 p - r_6 q \\
\dfrac{d\beta_2}{dx} - \dfrac{d\alpha_2}{dy} = p\dfrac{d\alpha_3}{dy} - q\dfrac{d\alpha_3}{dx} = r_{14} p - r_{10} q \\
\dfrac{d\gamma_0}{dx} - \dfrac{d\beta_0}{dy} = p\dfrac{d\beta_1}{dy} - q\dfrac{d\beta_1}{dx} = r_8 p - r_5 q \\
\dfrac{d\gamma_1}{dx} - \dfrac{d\beta_1}{dy} = p\dfrac{d\beta_2}{dy} - q\dfrac{d\beta_2}{dx} = r_{13} p - r_9 q \\
\dfrac{d\delta_0}{dx} - \dfrac{d\gamma_0}{dy} = p\dfrac{d\gamma_1}{dy} - q\dfrac{d\gamma_1}{dx} = r_{12} p - r_8 q
\end{array}\right\} .$$

These six identical relations reduce the twenty equations in the two foregoing columns to fourteen independent equations; hence the fifteen derivatives r can be expressed in terms of one of them, say r_5, and of derivatives of $\alpha_0, \ldots, \delta_0$; and the value of r_5 is $d\beta_0/du \div dz/du$. These expressions for the other fourteen, in terms of r_5, are

$$\left.\begin{aligned}
r_1 &= -r_5\frac{p^2}{q} + \frac{p}{q}\left(p\frac{d\alpha_1}{dy} - q\frac{d\alpha_1}{dx}\right) + \frac{d\alpha_0}{dx} \\
r_2 &= -r_5 p + \frac{d\beta_0}{dx} \\
r_3 &= r_5\frac{p}{q} - \frac{1}{q}\left(p\frac{d\alpha_1}{dy} - q\frac{d\alpha_1}{dx}\right) \\
r_4 &= -r_5 q + \frac{d\beta_0}{dy} \\
r_6 &= -r_5\frac{1}{q} + \frac{1}{q}\frac{d\alpha_1}{dy} \\
r_7 &= -r_5\frac{q^2}{p} + \frac{q}{p}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) + \frac{d\gamma_0}{dy} \\
r_8 &= r_5\frac{q}{p} - \frac{1}{p}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) \\
r_9 &= -r_5\frac{1}{p} + \frac{1}{p}\frac{d\beta_1}{dx} \\
r_{10} &= r_5\frac{1}{pq} - \frac{1}{pq}\frac{d\alpha_1}{dy} + \frac{1}{p}\frac{d\alpha_2}{dx} \\
r_{11} &= -r_5\frac{q^3}{p^2} + \frac{q^2}{p^2}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) + \frac{q}{p}\left(q\frac{d\gamma_1}{dx} - p\frac{d\gamma_1}{dy}\right) + \frac{d\delta_0}{dy} \\
r_{12} &= r_5\frac{q^2}{p^2} - \frac{q}{p^2}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) - \frac{1}{p}\left(q\frac{d\gamma_1}{dx} - p\frac{d\gamma_1}{dy}\right) \\
r_{13} &= -r_5\frac{q}{p^2} + \frac{1}{p^2}\left(q\frac{d\beta_1}{dx} - p\frac{d\beta_1}{dy}\right) + \frac{1}{p}\frac{d\gamma_1}{dx} \\
r_{14} &= r_5\frac{1}{p^2} - \frac{1}{p^2}\frac{d\beta_1}{dx} + \frac{1}{p}\frac{d\beta_2}{dx} \\
r_{15} &= -r_5\frac{1}{qp^2} + \frac{1}{qp^2}\frac{d\beta_1}{dx} - \frac{1}{qp}\frac{d\beta_2}{dx} + \frac{1}{q}\frac{d\alpha_3}{dy}
\end{aligned}\right\}.$$

20. When these values are substituted in the six equations, and terms are collected, it appears that the terms in r_5 in the six equations are

$$-r_5\frac{\Delta}{q}, \quad -r_5\frac{\Delta}{p}, \quad r_5\frac{\Delta}{pq}, \quad -r_5\frac{\Delta q}{p^2}, \quad r_5\frac{\Delta}{p^2}, \quad -r_5\frac{\Delta}{p^2 q},$$

respectively. For reasons which, being the same as before, need not be repeated here, the term in r_5 is made to disappear from each of the equations, and thus we have

$$\mathrm{A}p^2 + \mathrm{H}pq + \mathrm{B}q^2 - \mathrm{G}p - \mathrm{F}q + \mathrm{C} = 0,$$

the characteristic invariant. (We hence notice that this is a particular illustration of the remark in § 16, viz., if two equations are compatible with one another, their characteristic invariants are either the same or, being resoluble, have at least one factor common.)

The six equations, after some reductions of an easy character, take the form

$$(\mathrm{XX}) + \mathrm{A}\left(\frac{d\alpha_0}{dx} - p\frac{d\alpha_1}{dx}\right) + \mathrm{H}\left(\frac{d\beta_0}{dx} - p\frac{d\alpha_1}{dy}\right) + \mathrm{B}\left(\frac{d\beta_0}{dy} - q\frac{d\alpha_1}{dy}\right) + \mathrm{G}\frac{d\alpha_1}{dx} + \mathrm{F}\frac{d\alpha_1}{dy} = 0,$$

$$(\mathrm{XY}) + \mathrm{A}\left(\frac{d\beta_0}{dx} - p\frac{d\beta_1}{dx}\right) + \mathrm{H}\left(\frac{d\gamma_0}{dx} - p\frac{d\beta_1}{dy}\right) + \mathrm{B}\left(\frac{d\gamma_0}{dy} - q\frac{d\beta_1}{dy}\right) + \mathrm{G}\frac{d\beta_1}{dx} + \mathrm{F}\frac{d\beta_1}{dy} = 0,$$

$$(\mathrm{XZ}) + \mathrm{A}\left(\frac{d\alpha_1}{dx} - p\frac{d\alpha_2}{dx}\right) + \mathrm{H}\left(\frac{d\beta_1}{dx} - p\frac{d\alpha_2}{dy}\right) + \mathrm{B}\left(\frac{d\beta_1}{dy} - q\frac{d\alpha_2}{dy}\right) + \mathrm{G}\frac{d\alpha_2}{dx} + \mathrm{F}\frac{d\alpha_2}{dy} = 0,$$

$$(\mathrm{YY}) + \mathrm{A}\left(\frac{d\gamma_0}{dx} - p\frac{d\gamma_1}{dx}\right) + \mathrm{H}\left(\frac{d\delta_0}{dx} - p\frac{d\gamma_1}{dy}\right) + \mathrm{B}\left(\frac{d\delta_0}{dy} - q\frac{d\gamma_1}{dy}\right) + \mathrm{G}\frac{d\gamma_1}{dx} + \mathrm{F}\frac{d\gamma_1}{dy} = 0,$$

$$(\mathrm{YZ}) + \mathrm{A}\left(\frac{d\beta_1}{dx} - p\frac{d\beta_2}{dx}\right) + \mathrm{H}\left(\frac{d\gamma_1}{dx} - p\frac{d\beta_2}{dy}\right) + \mathrm{B}\left(\frac{d\gamma_1}{dy} - q\frac{d\beta_2}{dy}\right) + \mathrm{G}\frac{d\beta_2}{dx} + \mathrm{F}\frac{d\beta_2}{dy} = 0,$$

$$(\mathrm{ZZ}) + \mathrm{A}\left(\frac{d\alpha_2}{dx} - p\frac{d\alpha_3}{dx}\right) + \mathrm{H}\left(\frac{d\beta_2}{dx} - p\frac{d\alpha_3}{dy}\right) + \mathrm{B}\left(\frac{d\beta_2}{dy} - q\frac{d\alpha_3}{dy}\right) + \mathrm{G}\frac{d\alpha_3}{dx} + \mathrm{F}\frac{d\alpha_3}{dy} = 0.$$

For the aggregate of differential equations in the system, we have one pair

$$\frac{dv}{dx} = l + np, \qquad \frac{dv}{dy} = m + nq\,;$$

three pairs, one of which is

$$\frac{dl}{dx} = a + gp, \qquad \frac{dl}{dy} = h + gp\,;$$

six pairs, one of which is

$$\frac{da}{dx} = \alpha_0 + \alpha_1 p, \qquad \frac{da}{dy} = \beta_0 + \alpha_1 q\,;$$

one characteristic equation,

$$\Delta = 0\,;$$

and the above six equations.

The three equations of § 18 become three integrable combinations of the above six with the foregoing equations in the six pairs.

The number of quantities to be determined is

10 quantities	$\alpha_0, \ldots, \delta_0$;
6 ,, ...	$a, \ldots, h$;
3 ,, ...	l, m, n ;
1 quantity	v ;
1 ,, . . .	z ;

or 21 in all. There are certain relations among the differential equations; and further, four integrable combinations of the new system are known to exist, viz., the initial equation $F = 0$ and the three equations derived from it. What is wanted is, if existing, a new integrable combination.

21. Thus far the analysis applies whether the characteristic invariant is or is not reducible to two linear equations. Suppose now that, as in § 15, it vanishes, in virtue of one or other of the two equations

$$\left.\begin{aligned} Ap + (\tfrac{1}{2}H - \theta)\,q - (\tfrac{1}{2}G - \theta\phi) = 0 \\ Ap + (\tfrac{1}{2}H + \theta)\,q - (\tfrac{1}{2}G + \theta\phi) = 0 \end{aligned}\right\};$$

and consider these in turn, in the same manner as before.

Then, by combining the equations of identity with the equations particular to $F = 0$, we have

$$\left.\begin{aligned}
(XX) + A\delta\alpha_0 + (\tfrac{1}{2}H + \theta)\,\delta\beta_0 + (\tfrac{1}{2}G + \theta\phi)\,\delta\alpha_1 = 0 \\
(XY) + A\delta\beta_0 + (\tfrac{1}{2}H + \theta)\,\delta\gamma_0 + (\tfrac{1}{2}G + \theta\phi)\,\delta\beta_1 = 0 \\
(XZ) + A\delta\alpha_1 + (\tfrac{1}{2}H + \theta)\,\delta\beta_1 + (\tfrac{1}{2}G + \theta\phi)\,\delta\alpha_2 = 0 \\
(YY) + A\delta\gamma_0 + (\tfrac{1}{2}H + \theta)\,\delta\delta_0 + (\tfrac{1}{2}G + \theta\phi)\,\delta\gamma_1 = 0 \\
(YZ) + A\delta\beta_1 + (\tfrac{1}{2}H + \theta)\,\delta\gamma_1 + (\tfrac{1}{2}G + \theta\phi)\,\delta\beta_2 = 0 \\
(ZZ) + A\delta\alpha_2 + (\tfrac{1}{2}H + \theta)\,\delta\beta_2 + (\tfrac{1}{2}G + \theta\phi)\,\delta\alpha_3 = 0
\end{aligned}\right\};$$

as a modified form of the equations for

$$Ap + (\tfrac{1}{2}H - \theta)\,q - (\tfrac{1}{2}G - \theta\phi) = 0;$$

and in this form

$$\delta = \frac{d}{dx} + \frac{\tfrac{1}{2}H - \theta}{A}\,\frac{d}{dy}.$$

Let now

$$E(\alpha_0, \ldots, \delta_0, \ldots, h, l, m, n, v, x, y, z) = 0$$

be an integrable combination, so that we must have

$$\Sigma\,\frac{\partial E}{\partial\alpha_0}\,\frac{d\alpha_0}{dx} + \Sigma\,\frac{\partial E}{\partial a}\,\frac{da}{dx} + \Sigma\,\frac{\partial E}{\partial l}\,\frac{dl}{dx} + \frac{\partial E}{\partial v}\,(l + np) + \frac{\partial E}{\partial x} + p\,\frac{\partial E}{\partial z} = 0,$$

$$\Sigma\,\frac{\partial E}{\partial\alpha_0}\,\frac{d\alpha_0}{dy} + \Sigma\,\frac{\partial E}{\partial a}\,\frac{da}{dy} + \Sigma\,\frac{\partial E}{\partial l}\,\frac{dl}{dy} + \frac{\partial E}{\partial v}\,(m + nq) + \frac{\partial E}{\partial y} + q\,\frac{\partial E}{\partial z} = 0.$$

Multiply the first by A, the second by $\tfrac{1}{2}H - \theta$, and add; then using the equation connecting p and q only, we find

$$A\Sigma\,\frac{\partial E}{\partial\alpha_0}\,\delta\alpha_0 + A\,\frac{DE}{Dx} + (\tfrac{1}{2}H - \theta)\,\frac{DE}{Dy} + (\tfrac{1}{2}G - \theta\phi)\,\frac{DE}{Dz} = 0,$$

where

$$\left.\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}x} &= \frac{\partial}{\partial x} + l\frac{\partial}{\partial v} + a\frac{\partial}{\partial l} + h\frac{\partial}{\partial m} + g\frac{\partial}{\partial n}\\
&\qquad + \alpha_0\frac{\partial}{\partial a} + \beta_0\frac{\partial}{\partial h} + \alpha_1\frac{\partial}{\partial g} + \gamma_0\frac{\partial}{\partial b} + \beta_1\frac{\partial}{\partial f} + \alpha_2\frac{\partial}{\partial c}\\
\frac{\mathrm{D}}{\mathrm{D}y} &= \frac{\partial}{\partial y} + m\frac{\partial}{\partial v} + h\frac{\partial}{\partial l} + b\frac{\partial}{\partial m} + f\frac{\partial}{\partial n}\\
&\qquad + \beta_0\frac{\partial}{\partial a} + \gamma_0\frac{\partial}{\partial h} + \beta_1\frac{\partial}{\partial g} + \delta_0\frac{\partial}{\partial b} + \gamma_1\frac{\partial}{\partial f} + \beta_2\frac{\partial}{\partial c}\\
\frac{\mathrm{D}}{\mathrm{D}z} &= \frac{\partial}{\partial z} + n\frac{\partial}{\partial v} + g\frac{\partial}{\partial l} + f\frac{\partial}{\partial m} + c\frac{\partial}{\partial n}\\
&\qquad + \alpha_1\frac{\partial}{\partial a} + \beta_1\frac{\partial}{\partial h} + \alpha_2\frac{\partial}{\partial g} + \gamma_1\frac{\partial}{\partial b} + \beta_2\frac{\partial}{\partial f} + \alpha_3\frac{\partial}{\partial c}
\end{aligned}\right\}.$$

Now this is to be satisfied in virtue of the preceding six equations and independently of the particular values of $\partial\alpha_0, \ldots, \partial\delta_0$, belonging to any integral of the original equation. Hence the equation must be expressible in the form

$$\begin{gathered}
\lambda_1\{(\mathrm{XX}) + \ldots\} + \lambda_2\{(\mathrm{XY}) + \ldots\} + \lambda_3\{(\mathrm{XZ}) + \ldots\}\\
+ \lambda_4\{(\mathrm{YY}) + \ldots\} + \lambda_5\{(\mathrm{YZ}) + \ldots\} + \lambda_6\{(\mathrm{ZZ}) + \ldots\},
\end{gathered}$$

where $\lambda_1, \ldots, \lambda_6$, are indeterminate multipliers; the conditions sufficient and necessary for this are

$$(\tfrac{1}{2}\mathrm{H} + \theta)^3\frac{\partial\mathrm{E}}{\partial\alpha_0} - \mathrm{A}(\tfrac{1}{2}\mathrm{H} + \theta)^2\frac{\partial\mathrm{E}}{\partial\beta_0} + \mathrm{A}^2(\tfrac{1}{2}\mathrm{H} + \theta)\frac{\partial\mathrm{E}}{\partial\gamma_0} - \mathrm{A}^3\frac{\partial\mathrm{E}}{\partial\delta_0} = 0,$$

$$(\tfrac{1}{2}\mathrm{G} + \theta\phi)^3\frac{\partial\mathrm{E}}{\partial\alpha_0} - \mathrm{A}(\tfrac{1}{2}\mathrm{G} + \theta\phi)^2\frac{\partial\mathrm{E}}{\partial\alpha_1} + \mathrm{A}^2(\tfrac{1}{2}\mathrm{G} + \theta\phi)\frac{\partial\mathrm{E}}{\partial\alpha_2} - \mathrm{A}^3\frac{\partial\mathrm{E}}{\partial\alpha_3} = 0,$$

$$\begin{aligned}
(\tfrac{1}{2}\mathrm{G} + \theta\phi)^3\frac{\partial\mathrm{E}}{\partial\delta_0} &- (\tfrac{1}{2}\mathrm{G} + \theta\phi)^2(\tfrac{1}{2}\mathrm{H} + \theta)\frac{\partial\mathrm{E}}{\partial\gamma_1}\\
&+ (\tfrac{1}{2}\mathrm{G} + \theta\phi)(\tfrac{1}{2}\mathrm{H} + \theta)^2\frac{\partial\mathrm{E}}{\partial\beta_2} - (\tfrac{1}{2}\mathrm{H} + \theta)^3\frac{\partial\mathrm{E}}{\partial\alpha_3} = 0,
\end{aligned}$$

$$\begin{aligned}
2\frac{(\tfrac{1}{2}\mathrm{H} + \theta)(\tfrac{1}{2}\mathrm{G} + \theta\phi)}{\mathrm{A}}\frac{\partial\mathrm{E}}{\partial\alpha_0} &+ \frac{\mathrm{A}^2(\tfrac{1}{2}\mathrm{H} + \theta)}{(\tfrac{1}{2}\mathrm{G} + \theta\phi)^2}\frac{\partial\mathrm{E}}{\partial\alpha_3} + \mathrm{A}\frac{\partial\mathrm{E}}{\partial\beta_1}\\
&- (\tfrac{1}{2}\mathrm{G} + \theta\phi)\frac{\partial\mathrm{E}}{\partial\beta_0} - (\tfrac{1}{2}\mathrm{H} + \theta)\frac{\partial\mathrm{E}}{\partial\alpha_1} - \frac{\mathrm{A}^2}{\tfrac{1}{2}\mathrm{G} + \theta\phi}\frac{\partial\mathrm{E}}{\partial\beta_2} = 0,
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{XX})\frac{\partial\mathrm{E}}{\partial\alpha_0} &+ (\mathrm{XY})\left\{\frac{\partial\mathrm{E}}{\partial\beta_0} - \frac{\tfrac{1}{2}\mathrm{H} + \theta}{\mathrm{A}}\frac{\partial\mathrm{E}}{\partial\alpha_0}\right\} + (\mathrm{XZ})\left\{\frac{\partial\mathrm{E}}{\partial\alpha_1} - \frac{\tfrac{1}{2}\mathrm{G} + \theta\phi}{\mathrm{A}}\frac{\partial\mathrm{E}}{\partial\alpha_0}\right\} + (\mathrm{YY})\frac{\mathrm{A}}{\tfrac{1}{2}\mathrm{H} + \theta}\frac{\partial\mathrm{E}}{\partial\delta_0}\\
&+ (\mathrm{YZ})\left\{\frac{\mathrm{A}}{\tfrac{1}{2}\mathrm{G} + \theta\phi}\frac{\partial\mathrm{E}}{\partial\beta_2} - \frac{\mathrm{A}(\tfrac{1}{2}\mathrm{H} + \theta)}{(\tfrac{1}{2}\mathrm{G} + \theta\phi)^2}\frac{\partial\mathrm{E}}{\partial\alpha_3}\right\} + (\mathrm{ZZ})\frac{\mathrm{A}}{\tfrac{1}{2}\mathrm{G} + \theta\phi}\frac{\partial\mathrm{E}}{\partial\alpha_3} = 0.
\end{aligned}$$

22. This system must be rendered complete by the addition of the Jacobi-Poisson

conditions. If, when complete, the system contains N equations, then it possesses $23 - \mathrm{N}$ functionally independent solutions. Among these are to be included—

(i) The original differential equation $\mathrm{F} = 0$;

(ii) The three derivatives of $\mathrm{F} = 0$ with regard to x, y, z, respectively;

(iii) The two distinct integrals of

$$\frac{dx}{\mathrm{A}} = \frac{dy}{\frac{1}{2}\mathrm{H} - \theta} = \frac{dz}{\frac{1}{2}\mathrm{G} - \theta\phi};$$

say these are ξ, η.

Putting these on one side, there are thus $17 - \mathrm{N}$ new functionally independent solutions, so that N must be not greater than 16 in order that the method may be effective. If, when $\mathrm{N} = 16$, the solution is u, then

$$u = \psi(\xi, \eta),$$

where ψ is arbitrary, is an equation of the third order that can be associated with the given equation.

The same process, with corresponding results when the appropriate conditions are satisfied, is adopted for the alternative linear equation

$$\mathrm{A}p + (\tfrac{1}{2}\mathrm{H} + \theta)\,q - (\tfrac{1}{2}\mathrm{G} + \theta\phi)$$

arising out of the reducible characteristic invariant.

23. An example in which no equation of the first order involving only one arbitrary function, or no equation of the second order involving only one arbitrary function, can be associated with a given equation of the second order, is furnished by

$$a - h - g + f + 2\,\frac{2l - m - n}{y + z} = 0.$$

The general primitive is

$$\begin{aligned} v = \mathrm{F} + \mathrm{G} &+ \tfrac{1}{2}(y + z)\,\{\mathrm{F}_1 - \mathrm{F}_2 + \mathrm{G}_1 - \mathrm{G}_2\} \\ &+ \tfrac{1}{12}(y + z)^2\,\{\mathrm{F}_{11} - 2\mathrm{F}_{12} + \mathrm{F}_{22} + \mathrm{G}_{11} - 2\mathrm{G}_{12} + \mathrm{G}_{22}\}, \end{aligned}$$

where

$$\mathrm{F} = \mathrm{F}(x + y, z), \qquad \mathrm{G} = \mathrm{G}(x + z, y),$$

and the subscripts 1, 2, denote derivation with respect to the first and the second of the arguments in the respective cases. The associable equations are of the third order at lowest; and they are

$$\begin{aligned} \alpha_0 - 3\beta_0 + 3\gamma_0 - \delta_0 &= (y + z)\,\Phi(x + z, y), \\ \alpha_0 - 3\alpha_1 + 3\alpha_2 - \alpha_3 &= (y + z)\,\Psi(x + y, z), \end{aligned}$$

where Φ and Ψ are arbitrary.

In a similar way in part, and by induction in part, it may be proved that the integral of

$$a - h - g + f + \mathrm{I}\,\frac{2l - m - n}{y + z} = 0,$$

where I is a positive integer, can be expressed in finite terms. To express the integral, let

$$\mathrm{F}(\alpha, \beta) = \mathrm{F}, \qquad \text{where} \qquad \alpha = x + y, \beta = z,$$
$$\mathrm{G}(\alpha', \beta') = \mathrm{G}, \qquad \text{where} \qquad \alpha' = x + z, \beta' = y;$$

and denote by Δ the operation $\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial\beta}$, by Δ' the operation $\frac{\partial}{\partial\alpha'} - \frac{\partial}{\partial\beta'}$, so that

$$\Delta\mathrm{F} = \frac{\partial\mathrm{F}}{\partial\alpha} - \frac{\partial\mathrm{F}}{\partial\beta}, \qquad \Delta^2\mathrm{F} = \frac{\partial^2\mathrm{F}}{\partial\alpha^2} - 2\frac{\partial^2\mathrm{F}}{\partial\alpha\,\partial\beta} + \frac{\partial^2\mathrm{F}}{\partial\beta^2}, \ldots$$
$$\Delta'\mathrm{G} = \frac{\partial\mathrm{G}}{\partial\alpha'} - \frac{\partial\mathrm{G}}{\partial\beta'}, \qquad \Delta'^2\mathrm{G} = \frac{\partial^2\mathrm{G}}{\partial\alpha'^2} - 2\frac{\partial^2\mathrm{G}}{\partial\alpha'\,\partial\beta'} + \frac{\partial^2\mathrm{G}}{\partial\beta'^2}, \ldots.$$

Then the value of v is

$$\begin{aligned} v = {} & \mathrm{F} + \mathrm{G} \\ & + \tfrac{1}{2}(y+z)(\Delta\mathrm{F} + \Delta'\mathrm{G}) \\ & + \ldots\ldots\ldots\ldots \\ & + \frac{\mathrm{I}!\,(2\mathrm{I}-s)!}{(\mathrm{I}-s)!\,s!\,2\mathrm{I}!}(y+z)^s(\Delta^s\mathrm{F} + \Delta'^s\mathrm{G}) \\ & \ldots\ldots\ldots\ldots \\ & + \frac{\mathrm{I}!}{2\mathrm{I}!}(y+z)^{\mathrm{I}}(\Delta^{\mathrm{I}}\mathrm{F} + \Delta'^{\mathrm{I}}\mathrm{G}). \end{aligned}$$

24. But it is necessary to take account of what has been achieved when one equation or when two equations (say of the second order) have been obtained compatible with the given equation and involving each one arbitrary function. The method adopted in § 11 to pass to the primitive has manifestly no element of generality.

Now the three equations are not sufficient to express a, b, c, f, g, h, in terms of l, m, n, and the variables; but they frequently will serve to express groups of combinations of a, b, c, f, g, h, in terms of those quantities. Thus the three equations in § 11 suggest combinations $a - h$, $h - b$, $g - f$ (which are the derivatives of $l - m$), and $a - g$, $h - f$, $g - c$ (which are the derivatives of $l - n$). This, however, is only a slight modification of the former method; it, again, has no element of generality.

Another plan would be to differentiate the three equations up to any order with the hope of determining all the derivatives of the highest order that occur in terms of derivatives of lower order. If this were possible, substitution in the equations of differential elements such as

$$dl = a\,dx + h\,dy + g\,dz$$

and successive integration would ultimately lead to v. It appears in general, however, that relations of interdependence among the equations prevents them from being adequate for the purpose at any stage; the relations are, in fact, satisfied conditions of compatibility. This method is, therefore, ineffective.

An effective method can, however, be obtained as follows. Restricting ourselves for the moment to the equation of the second order with two compatible equations also of that order*—the restriction is made only to simplify the explanations—we have v as expressible in terms of two arbitrary functions. Hence each of the quantities l, m, n (and therefore any combination of v, l, m, n), can be expressed in terms of two arbitrary functions. Now in one of the compatible equations we have one arbitrary function which is to be identified with one of the arbitrary functions in v; hence it is to be expected that a proper combination of v, l, m, n, is an intermediary integral of that equation involving a new arbitrary function, which must be identified with the other of the arbitrary functions in v.

Similarly for the other of the compatible equations, there is an intermediary integral involving the two arbitrary functions. The conditions of coexistence of the two intermediary integrals must be assigned; it will appear that, if the conditions are not satisfied identically, they provide the means of identification of the various arbitrary functions.

It is to be observed that the intermediary integral or integrals thus obtained cannot be regarded as intermediary integrals of the original equation in the ordinary sense of the phrase, for each of them involves two arbitrary functions. But they are intermediary for the respective compatible equations: each of them involves one arbitrary function more than occurs in the compatible equation. The result manifestly does not imply that the original equation possesses any intermediary integral; in fact, the assumption throughout our investigations has been that no proper intermediary integral exists.

25. A method† has been given elsewhere for constructing the intermediary integral. In effect, it amounts to the use of the conditions which must be satisfied in order that the derivatives

$$\left.\begin{array}{r} au_l + hu_m + gu_n + u_x = 0 \\ hu_l + bu_m + fu_n + u_y = 0 \\ gu_l + fu_m + cu_n + u_z = 0 \end{array}\right\}$$

from the supposed integral

$$u\,(l, m, n, v, x, y, z) = 0$$

shall cause the compatible equation

$$\Theta\,(a, \ldots, h, l, m, n, v, x, y, z) = 0$$

to be satisfied without regard to the values of the differential coefficients of v of the second order. These conditions are the simultaneous partial differential equations of the first order determining u.

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* The explanations will be seen to apply, *mutatis mutandis*, to other cases of the second order, and indeed to cases of any order, when compatible equations are known.

† In the memoir cited in the introductory remarks.

Thus, dealing with the case of § 11, when the compatible equations of the second order are

$$a + f - g - h = -\frac{2l - m - n}{y + z},$$

$$a - 2h + b = (y + z)\,\theta\,(x + z, y),$$

$$a - 2g + c = (y + z)\,\phi\,(x + y, z),$$

we know that the first has no intermediary integral. As regards an intermediary for the second, substituting for a, b, c, from the derivatives of the u-equation, the result

$$-\left(\frac{u_m}{u_l}h + \frac{u_n}{u_l}g + \frac{u_x}{u_l}\right) - 2h - \left(\frac{u_l}{u_m}h + \frac{u_n}{u_m}f + \frac{u_y}{u_m}\right) = (y + z)\,\theta$$

must, *quâ* equation in f, g, h, be evanescent; hence we find

$$(u_m + u_l)^2 = 0, \qquad u_n = 0, \qquad -\frac{u_x}{u_l} - \frac{u_y}{u_m} = (y + z)\,\theta\,;$$

that is, the equations for u are

$$\frac{\partial u}{\partial n} = 0,$$

$$\frac{\partial u}{\partial l} + \frac{\partial u}{\partial m} = 0,$$

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} + (l - m)\frac{\partial u}{\partial v} - (y + z)\frac{\partial u}{\partial m}\,\theta = 0.$$

The system is a complete system; hence it possesses four functionally independent solutions. Writing

$$\theta = g_{111} - 3g_{112} + 3g_{122} - g_{222},$$

these four solutions can be expressed in the form

$$z,\ x + y,$$

$$l - m - (y + z)(g_{11} - 2g_{12} + g_{22}) - g_1 + g_2,$$

and another involving v, which would require either the arbitrary constant or the arbitrary functional form to which the third would be equated. (The fourth is, in fact, a primitive of the compatible equation under discussion, though it is not necessarily the common primitive of the three simultaneous equations.) We thus infer that

$$l - m - (y + z)(g_{11} - 2g_{12} + g_{22}) - g_1 + g_2 = \psi\,(x + y, z)$$

is an intermediary of the second of the equations.

In a similar manner, by writing

$$\phi = f_{111} - 3f_{112} + 3f_{122} - f_{222},$$

it can be shown that

$$l - n - (y + z)(f_{11} - 2f_{12} + f_{22}) - f_1 + f_2 = \chi(z + x, y)$$

is an intermediary of the third of the equations. When the conditions of coexistence of these two are assigned, they determine the arbitrary functions ψ and χ in the forms

$$\psi = -f_1 + f_2, \qquad \chi = -g_1 + g_2;$$

so that we have

$$\left.\begin{aligned} l - m &= (y + z)(g_{11} - 2g_{12} + g_{22}) + g_1 - g_2 - f_1 + f_2 \\ l - n &= (y + z)(f_{11} - 2f_{12} + f_{22}) + f_1 - f_2 - g_1 + g_2 \end{aligned}\right\};$$

and the primitive can be obtained by the customary process, leading to the form

$$v = 2f + 2g + (y + z)(f_1 - f_2 + g_1 - g_2),$$

where $f = f(x + y, z)$ and $g = g(x + z, y)$ are arbitrary functions.

26. If one or both of the equations compatible with the original equation were of the third order, we should then seek an equation of the second order involving one arbitrary function more than that equation of the third order; and we should proceed in a manner similar to that of the preceding plan, the conditions of coexistence of the different equations furnishing the means of identification or comparison of the arbitrary functions that occur.

If there be no equation of the third order, we should similarly proceed to obtain possible equations of the fourth order, if any; and so on with the orders in succession. The method is one of general application if equations of any order compatible with the original equation exist.

SECTION III.

Equations having an irresoluble characteristic invariant.

27. The investigations contained in the preceding sections of this paper have referred for the most part to those equations

$$F(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0,$$

whose characteristic invariant

$$p^2\frac{\partial F}{\partial a} + pq\frac{\partial F}{\partial h} + q^2\frac{\partial F}{\partial b} - p\frac{\partial F}{\partial g} - q\frac{\partial F}{\partial f} + \frac{\partial F}{\partial c} = 0$$

is resoluble into equations that are linear in p and q. Those contained in the present section refer to equations whose characteristic invariant is irresoluble.

In the first section (§ 2), a generalisation of AMPÈRE's method was dealt with very briefly, partly because that method and DARBOUX's method apply most effectively to equations in which (with few exceptions) the derivatives of the highest order occur linearly; and, of the two, it is DARBOUX's method which can be more effectively applied to other equations. The fact that the characteristic invariant was resoluble proved of material importance in the general theory.

It is to be remarked, however, that some of the equations which occur most frequently in mathematical physics, for example

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0,$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \mu \frac{\partial v}{\partial t},$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = c^2 \frac{\partial^2 v}{\partial t^2},$$

the latter two being, for purposes of application, made to depend upon the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -\kappa^2 v,$$

belong to the class which have their characteristic invariant not resoluble, and at the same time are linear in the derivatives of the highest orders that occur. Accordingly both AMPÈRE's method and DARBOUX's method generalised can be applied to such equations.

Moreover, the generalisation of AMPÈRE's method can also be applied to equations of the form

$$\theta\Theta + \mathrm{A}\frac{\partial\Theta}{\partial a} + \mathrm{B}\frac{\partial\Theta}{\partial b} + \mathrm{C}\frac{\partial\Theta}{\partial c} + \mathrm{F}\frac{\partial\Theta}{\partial f} + \mathrm{G}\frac{\partial\Theta}{\partial g} + \mathrm{H}\frac{\partial\Theta}{\partial h}$$
$$+ \mathrm{A}_1 a + \mathrm{B}_1 b + \mathrm{C}^1 c + \mathrm{F}_1 f + \mathrm{G}_1 g + \mathrm{H}_1 h = \mathrm{U},$$

where

$$\Theta = \begin{vmatrix} a, & h, & g, \\ h, & b, & f, \\ g, & f, & c, \end{vmatrix}$$

and the quantities θ, A, . . ., H, A_1, . . ., H_1, U, do not involve derivatives of the second order. For, when the equation is transformed by the relations of § 1, it takes the form

$$\mathrm{J} + c\mathrm{I} = 0,$$

where $\mathrm{I} = 0$ is the characteristic equation; in other words, taking account of $\mathrm{I} = 0$, we must associate $\mathrm{J} = 0$ with it as an equivalent to the postulated equation.

28. We consequently begin with the generalisation of AMPÈRE'S method. Let the variables be changed from x, y, z, to x, y, u, where u is a function of x, y, z, as yet undetermined, so that z is a function of x, y, u, as yet also undetermined. With the notation previously adopted, we have

$$\left.\begin{array}{lll}
\dfrac{dv}{dx} = l + np, & \dfrac{dv}{dy} = m + nq, & \dfrac{dv}{du} = n\dfrac{dz}{du} \\[2ex]
\dfrac{dl}{dx} = a + gp, & \dfrac{dl}{dy} = h + gq, & \dfrac{dl}{du} = g\dfrac{dz}{du} \\[2ex]
\dfrac{dm}{dx} = h + fp, & \dfrac{dm}{dy} = b + fq, & \dfrac{dm}{du} = f\dfrac{dz}{du} \\[2ex]
\dfrac{dn}{dx} = g + cp, & \dfrac{dn}{dy} = f + cq, & \dfrac{dn}{du} = c\dfrac{dz}{du}
\end{array}\right\};$$

and therefore, from the equations involving derivatives with regard to x and y alone, it follows that

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + p^2c, \qquad b = \frac{dm}{dy} - q\frac{dn}{dy} + q^2c,$$

$$g = \frac{dn}{dx} - pc, \qquad f = \frac{dn}{dy} - qc,$$

$$\left.\begin{aligned} h &= \frac{dm}{dx} - p\frac{dn}{dy} + pqc \\ &= \frac{dl}{dy} \quad q\frac{dn}{dx} + pqc \end{aligned}\right\},$$

so that we have

$$\frac{dm}{dx} - p\frac{dn}{dy} = \frac{dl}{dy} - q\frac{dn}{dx},$$

which is the condition in order that the necessary relation

$$\frac{d}{dy}\left(\frac{dv}{dx}\right) = \frac{d}{dx}\left(\frac{dv}{dy}\right)$$

be satisfied.

When the postulated equation of the second order is such that, on the substitution of the foregoing values for a, b, f, g, h, it has a linear form in c, let it be

$$J + cI = 0.$$

Suppose that the variable u (or z as a function of x, y, u) is determined so that

$$I = 0,$$

which, after the earlier explanations, is the characteristic invariant; then we have also

$$J = 0.$$

The system of equations now is

$$\left.\begin{array}{c} \mathrm{I} = 0, \qquad \mathrm{J} = 0 \\ \dfrac{dm}{dx} - p\dfrac{dn}{dy} = \dfrac{dl}{dy} - q\dfrac{dn}{dx} \\ \dfrac{dv}{dx} = l + np, \qquad \dfrac{dv}{dy} = m + nq \end{array}\right\},$$

involving the quantities l, m, n, v, z, as functions of x and y. In all the derivatives here contained, u is parametric; and consequently all the constants that arise in the integration are constants on this supposition: in other words, all of them are functions of u. Consequently, when the integrals of the system are obtained, one constant (at choice) can be taken to be u; all the other constants are then functions of u, arbitrary so far as the system is concerned; and any arbitrary function of x and y that occurs is also (possibly) a function of u. In order to determine the limitations on the arbitrary functions, the equation

$$\frac{dv}{du} = n\frac{dz}{du}$$

must also be satisfied; this equation will usually give relations among the arbitrary functional forms, or will determine one of them.

29. *The relations thus obtained constitute an integral of the equation.* For suppose that in the expression for v we consider u eliminated in favour of z; then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}\left(p\,dx + q\,dy + \frac{dz}{du}du\right).$$

But also

$$dv = \frac{dv}{dx}dx + \frac{dv}{dy}dy + \frac{dv}{du}du,$$

whence

$$\frac{dv}{du} = \frac{\partial v}{\partial z}\frac{dz}{du},$$

$$\frac{dv}{dx} = \frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z},$$

$$\frac{dv}{dy} = \frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z};$$

and therefore, comparing these with the equations of the system leading to the integral form, it follows that

$$l = \frac{\partial v}{\partial x}, \qquad m = \frac{\partial v}{\partial y}, \qquad n = \frac{\partial v}{\partial z}.$$

Next, take a quantity c such that

$$\frac{dn}{du} = c\frac{dz}{du}.$$

Since

$$\frac{dv}{dx} = l + np, \qquad \frac{dv}{du} = n\,\frac{dz}{du}$$

are satisfied by the integral relations, we have

$$\frac{d}{du}(l + np) = \frac{d}{dx}\left(n\,\frac{dz}{du}\right),$$

and therefore

$$\frac{dl}{du} + p\,\frac{dn}{du} = \frac{dn}{dx}\,\frac{dz}{du},$$

the terms in $\frac{d^2z}{dx\,du}$ cancelling; consequently

$$\frac{dl}{du} = \left(\frac{dn}{dx} \quad pc\right)\frac{dz}{du} = g\,\frac{dz}{du}.$$

Similarly, from

$$\frac{dv}{dy} = m + nq, \qquad \frac{dv}{du} = n\,\frac{dz}{du},$$

we find

$$\frac{dm}{du} = f\,\frac{dz}{du},$$

the quantities g and f which occur here being those which formally occur in the derivatives of l, m, n, with regard to x and y.

Again, we have

$$\frac{\partial^2 v}{\partial x^2}\,dx + \frac{\partial^2 v}{\partial x\,\partial y}\,dy + \frac{\partial^2 v}{\partial x\,\partial z}\left(p\,dx + q\,dy + \frac{dz}{du}\,du\right)$$

$$= d\,\frac{\partial v}{\partial x} = dl = \frac{dl}{dx}\,dx + \frac{dl}{dy}\,dy + \frac{dl}{du}\,du,$$

whence

$$\frac{dl}{du} = \frac{\partial^2 v}{\partial x\,\partial z}\,\frac{dz}{du},$$

$$\frac{dl}{dx} = \frac{\partial^2 v}{\partial x^2} + p\,\frac{\partial^2 v}{\partial x\,\partial z},$$

$$\frac{dl}{dy} = \frac{\partial^2 v}{\partial x\,\partial y} + q\,\frac{\partial^2 v}{\partial x\,\partial z};$$

and therefore, comparing with the former equations, we have

$$a = \frac{\partial^2 v}{\partial x^2}, \quad h = \frac{\partial^2 v}{\partial x\,\partial y}, \quad g = \frac{\partial^2 v}{\partial x\,\partial z}.$$

Similarly we find

$$b = \frac{\partial^2 v}{\partial y^2}, \quad f = \frac{\partial^2 v}{\partial y\,\partial z}, \quad c = \frac{\partial^2 v}{\partial z^2}.$$

Now, when we take the combination

$$\mathrm{J} + c\mathrm{I} = 0,$$

and eliminate the derivatives of l, m, n, with regard to x and y, we have the original equation

$$\mathrm{F} = 0,$$

or the original equation is satisfied in virtue of the integral system. But, from this integral system, the value of v is such that

$$l, m, n = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) v,$$

$$a, b, c, f, g, h = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}, \frac{\partial^2}{\partial y\,\partial z}, \frac{\partial^2}{\partial z\,\partial x}, \frac{\partial^2}{\partial x\,\partial y}\right) v;$$

and therefore v is an integral of the partial differential equation. We consequently have the theorem—

When an equation $\mathrm{F} = 0$ *of the second order is transformed into* $\mathrm{J} + c\mathrm{I} = 0$ *by means of the equations*

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + p^2 c, \quad b = \frac{dm}{dy} - q\frac{dn}{dy} + q^2 c,$$

$$g = \frac{dn}{dx} - pc, \qquad f = \frac{dn}{dy} - qc,$$

$$\left.\begin{aligned} h &= \frac{dm}{dx} - p\frac{dn}{dy} + pqc \\ &= \frac{dl}{dy} - q\frac{dn}{dx} + pqc \end{aligned}\right\},$$

and when, in the integral equivalent of the simultaneous system

$$\left.\begin{aligned} &\mathrm{I} = 0, \quad \mathrm{J} = 0 \\ &\frac{dm}{dx} - p\frac{dn}{dy} = \frac{dl}{dy} - q\frac{dn}{dx} \\ &\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq \end{aligned}\right\},$$

all the arbitrary constants are made functions of a parameter u, *and the arbitrary functions of* x *and* y *are also made functions of* u, *subject to the equation*

$$\frac{dv}{du} = n\frac{dz}{du},$$

(*which, in fact, will generally determine either an arbitrary function or relations among the arbitrary functions*), *then the value of* v *thus obtained is an integral of the original equation* $\mathrm{F} = 0$.

30. The integration of the equation $\mathrm{F} = 0$ is thus made to depend upon the integration of a simultaneous system involving fewer independent variables, and

upon the subsequent determination of the arbitrary functions in the integral equivalent of the system. A question therefore arises as to how an integral equivalent can be obtained. At first sight it seems that, as the number of equations (being five) is equal to the number of unknowns (l, m, n, v, z) to be determined, HAMBURGER'S method* might be applied to our special instance, though not when the number of independent variables in the original equation is more than three. But, as a matter of fact, one of the equations of the system is a functional consequence of two others; viz., the equation

$$\frac{dm}{dx} - p\frac{dn}{dy} = \frac{dl}{dy} - q\frac{dn}{dx}$$

is a functional consequence of

$$\frac{dv}{dx} = l + np, \quad \frac{dv}{dy} = m + nq.$$

It thus follows that there are only four equations independent of one another involving the five variables; consequently HAMBURGER'S method does not apply. On the other hand, the inference is that, as the equations are fewer in number by unity than the number of variables to be determined, one arbitrary element must exist in any general integral equivalent. This arbitrary element and other arbitrary functional forms, by the foregoing theory, are determined by means of the equation

$$\frac{dv}{du} = n\frac{dz}{du},$$

so far as they can be made determinate.

It is therefore necessary to seek for some integral combination of the subsidiary system, apparently without at present having any perfectly general process of constructing such a solution. It may, however, be pointed out that, as there are four independent equations involving five quantities, they can be used to determine four of them in terms of the remaining one or, more symmetrically when this is possible, to express all five of them in terms of some variable. When such expressions have been obtained, they are to be substituted in

$$\frac{dv}{du} = n\frac{dz}{du},$$

the full solution of the resulting form of which equation will then serve to determine the quantities.

We proceed to consider one or two examples in connection with the foregoing theory and explanations, dealing particularly with well-known equations.

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* CRELLE, t. lxxxi. (1876), pp. 243–281; *ib.*, t. xciii. (1882), pp. 188–214, the number of independent variables being two, and the number of equations being equal to the number of dependent variables.

Application to $\nabla^2 v = 0$.

31. When the method is applied to the potential equation, which is

$$a + b + c = 0$$

with the present notation, the substitution of values (say of a and b) is required to lead to a result evanescent so far as the determination of coefficients of the second order is concerned. The substitution gives

$$\frac{dl}{dx} - p\frac{dn}{dx} + \frac{dm}{dy} - q\frac{dn}{dy} + c\,(p^2 + q^2 + 1) = 0$$

so that we must have

$$p^2 + q^2 + 1 = 0,$$
$$\frac{dl}{dx} - p\frac{dn}{dx} + \frac{dm}{dy} - q\frac{dn}{dy} = 0\,;$$

and the differentiations with regard to x and to y in these relations are effected on the supposition that the unexpressed variable u is constant.

The subsidiary simultaneous system thus is

$$\left.\begin{aligned} p^2 + q^2 + 1 &= 0 \\ \frac{dl}{dx} - p\frac{dn}{dx} + \frac{dm}{dy} - q\frac{dn}{dy} &= 0 \\ \frac{dl}{dy} + p\frac{dn}{dy} - \frac{dm}{dx} - q\frac{dn}{dx} &= 0 \\ \frac{dv}{dx} &= l + np \\ \frac{dv}{dy} &= m + nq \end{aligned}\right\}$$

When Hamburger's method, as expounded in the second of his memoirs already quoted (§ 30), is applied to this system, it is found that the algebraical equations for the determination of the subsidiary multipliers are inconsistent with one another unless all the multipliers are zero; there is then a null result. Accordingly integrable combinations must be obtained otherwise.

Now the general solution of the equation

$$p^2 + q^2 + 1 = 0$$

is given by

$$p = \text{constant}, \qquad q = \text{constant},$$
$$z - px - qy = \text{constant},$$

these constants occurring in association with u constant. We may, therefore, assume

$$z = u + xp(u) + yq(u),$$

where p and q are arbitrary functions of u, subject solely to the condition

$$p^2 + q^2 + 1 = 0.$$

Because the differentiations with regard to x and to y are effected on the hypothesis that u is constant, the other equation can be taken in the form

$$\frac{d}{dx}(l - np) + \frac{d}{dy}(m - nq) = 0,$$

so that a function ξ of x and y (and possibly also involving u) exists such that

$$l - np = \frac{d\xi}{dy}, \quad m - nq = -\frac{d\xi}{dx}.$$

Moreover, we have

$$l + np = \frac{dv}{dx}, \quad m + nq = \frac{dv}{dy};$$

consequently

$$2l = \frac{dv}{dx} + \frac{d\xi}{dy},$$

$$2m = \frac{dv}{dy} - \frac{d\xi}{dx},$$

$$2np = \frac{dv}{dx} - \frac{d\xi}{dy}, \quad 2nq = \frac{dv}{dy} + \frac{d\xi}{dx}.$$

From the last two equations, it follows that

$$q\left(\frac{dv}{dx} - \frac{d\xi}{dy}\right) = p\left(\frac{dv}{dy} + \frac{d\xi}{dx}\right),$$

and thence that

$$\frac{d}{dx}(qv - p\xi) = \frac{d}{dy}(pv + q\xi).$$

Consequently a function w of x and y (and possibly also involving u) exists such that

$$pv + q\xi = -\frac{dw}{dx},$$

$$qv - p\xi = -\frac{dw}{dy},$$

whence

$$v = p\frac{dw}{dx} + q\frac{dw}{dy},$$

$$\xi = q\frac{dw}{dx} - p\frac{dw}{dy}.$$

Substituting these values, we have

$$\left.\begin{aligned} 2l &= \; p\frac{d^2w}{dx^2} + 2q\frac{d^2w}{dx\,dy} - p\frac{d^2w}{dy^2} \\ 2m &= -q\frac{d^2w}{dx^2} + 2p\frac{d^2w}{dx\,dy} + q\frac{d^2w}{dy^2} \\ 2n &= \;\frac{d^2w}{dx^2} \qquad\qquad\qquad + \frac{d^2w}{dy^2} \end{aligned}\right\}.$$

Thus far account of variations with regard to x and to y has been taken. But, as regards variations of u, we have

$$\frac{dv}{du} = n\frac{dz}{du},$$

that is,

$$2\frac{d}{du}\left(p\frac{dw}{dx} + q\frac{dw}{dy}\right) = \Delta\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2}\right),$$

where Δ denotes $1 + xp' + yq'$. This is the equation of limitation upon the form of w; if its general integral were known, the general value of v could be deduced.

2. Before proceeding with the consideration of this result, it is worth noting the relation of the equations

$$l - np = \frac{d\xi}{dy}, \qquad m - nq = -\frac{d\xi}{dx},$$

to the original equation. Because

$$z = u + xp + yq,$$

it follows that

$$\frac{1}{p}\frac{\partial u}{\partial x} = \frac{1}{q}\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial z} = -\frac{1}{\Delta},$$

where Δ denotes

$$1 + xp' + yq',$$

so that

$$\left.\begin{aligned} l - np &= \frac{\partial\xi}{\partial y} + q\frac{\partial\xi}{\partial z} \\ m - nq &= -\frac{\partial\xi}{\partial x} - p\frac{\partial\xi}{\partial z} \end{aligned}\right\},$$

when in ξ substitution for u is made in terms of x, y, z. From the former we have

$$\begin{aligned} a - gp - np'\frac{\partial u}{\partial x} &= \frac{\partial^2\xi}{\partial x\,\partial y} + q\frac{\partial^2\xi}{\partial x\,\partial z} + \frac{\partial\xi}{\partial z}q'\frac{\partial u}{\partial x} \\ g - cp - np'\frac{\partial u}{\partial z} &= \frac{\partial^2\xi}{\partial y\,\partial z} + q\frac{\partial^2\xi}{\partial z^2} + \frac{\partial\xi}{\partial z}q'\frac{\partial u}{\partial z}, \end{aligned}$$

and from the latter

$$b - fq - nq' \frac{\partial u}{\partial y} = - \frac{\partial^2 \xi}{\partial x \, \partial y} - p \frac{\partial^2 \xi}{\partial y \, \partial z} - \frac{\partial \xi}{\partial z} p' \frac{\partial u}{\partial y} ,$$

$$f - cq - nq' \frac{\partial u}{\partial z} = - \frac{\partial^2 \xi}{\partial x \, \partial z} - p \frac{\partial^2 \xi}{\partial z^2} - \frac{\partial \xi}{\partial z} p' \frac{\partial u}{\partial z} .$$

Multiplying the second of these equations by p, the fourth by q, and then adding all four, the function ξ is eliminated, and we have

$$a + b + c = 0.$$

33. Again, the general theory in a preceding Section (see particularly § 4) immediately suggests that u occurs as an argument of an arbitrary function. This being so, let the variables be considered as transformed to x, y, u, where u now is known, and effect the transformation directly upon the equation

$$a + b + c = 0.$$

We have, for any function P,

$$\begin{aligned} \frac{\partial \mathrm{P}}{\partial x} dx + \frac{\partial \mathrm{P}}{\partial y} dy + \frac{\partial \mathrm{P}}{\partial z} dz &= d\mathrm{P} \\ &= \frac{d\mathrm{P}}{dx} dx + \frac{d\mathrm{P}}{dy} dy + \frac{d\mathrm{P}}{du} du \\ &= \frac{d\mathrm{P}}{dx} dx + \frac{d\mathrm{P}}{dy} dy + \frac{d\mathrm{P}}{du} \frac{1}{\Delta} (dz - p \, dx - q \, dy), \end{aligned}$$

so that

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{d}{dx} - \frac{p}{\Delta} \frac{d}{du}, \\ \frac{\partial}{\partial y} &= \frac{d}{dy} - \frac{q}{\Delta} \frac{d}{du}, \\ \frac{\partial}{\partial z} &= \qquad\quad \frac{1}{\Delta} \frac{d}{du}. \end{aligned}$$

We thus have

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= \frac{d^2 v}{dx^2} - 2 \frac{p}{\Delta} \frac{d^2 v}{dx \, du} + \frac{p^2}{\Delta^2} \frac{d^2 v}{du^2} - \frac{dv}{du} \left\{ - \frac{2pp'}{\Delta^2} + \frac{p^2}{\Delta^3} (xp'' + yq'') \right\}, \\ \frac{\partial^2 v}{\partial y^2} &= \frac{d^2 v}{dy^2} - 2 \frac{q}{\Delta} \frac{d^2 v}{dy \, du} + \frac{q^2}{\Delta^2} \frac{d^2 v}{du^2} - \frac{dv}{du} \left\{ - \frac{2qq'}{\Delta} + \frac{q^2}{\Delta^3} (xp'' + yq'') \right\}, \\ \frac{\partial^2 v}{\partial z^2} &= \qquad\qquad\qquad\qquad\quad \frac{1}{\Delta^2} \frac{d^2 v}{du^2} - \frac{dv}{du} \left\{ \qquad\qquad\quad \frac{1}{\Delta^3} (xp'' + yq'') \right\}, \end{aligned}$$

whence, adding and remembering that $a + b + c = 0$, it follows that

$$\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} - \frac{2}{\Delta} \left(p \frac{d^2 v}{dx \, du} + q \frac{d^2 v}{dy \, du} \right) = 0.$$

It is immediately obvious that *one solution* of this equation (and therefore *of the original equation*) *is given by equating v to any arbitrary function of u*, a result that admits of simple verification.*

34. But it is not at first sight clear how this solution connects itself with the general solution indicated in § 31; the connection can be made as follows.

The general value of v is

$$v = p\frac{dw}{dx} + q\frac{dw}{dy};$$

if this is to be an arbitrary function of u, say $f(u)$, as for the solution under consideration, we must have

$$p\frac{dw}{dx} + q\frac{dw}{dy} = f(u),$$

and consequently

$$w = \frac{x}{p}f(u) + \mathrm{G}(u, p'x + q'y),$$

where, so far as concerns this relation, G is any arbitrary function of both its arguments. Writing

$$xp' + yq' = \eta,$$

this is

$$w = \frac{x}{p}f(u) + \mathrm{G}(u, \eta),$$

so that

$$\frac{d^2w}{dx^2} = p'^2\frac{\partial^2\mathrm{G}}{\partial\eta^2},$$

$$\frac{d^2w}{dx\,dy} = p'q'\frac{\partial^2\mathrm{G}}{\partial\eta^2},$$

$$\frac{d^2w}{dy^2} = q'^2\frac{\partial^2\mathrm{G}}{\partial\eta^2}.$$

Now

$$p^2 + q^2 + 1 = 0,$$

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* This result was published in the 'Messenger of Mathematics,' vol. xxvii. (1898), pp. 99–118, in a short paper entitled "New Solutions of some of the Partial Differential Equations of Mathematical Physics." The form was altered from that in the text, so that it might be symmetric in the variables, and the theorem was given as follows:—

If p, q, r, be three arbitrary functions of u such that

$$p^2 + q^2 + r^2 = 0,$$

and if u be determined as a function of x, y, z, by the equation

$$au = xp + yq + zr,$$

where a is any constant, also if v denote any arbitrary function of u, then v satisfies LAPLACE'S *equation*

$$\nabla^2 v = 0.$$

so that

$$pp' + qq' = 0,$$

say

$$\frac{p'}{q} = \frac{q'}{-p} = \theta\,;$$

and so

$$\frac{1}{q^2}\frac{d^2w}{dx^2} = -\frac{1}{pq}\frac{d^2w}{dx\,dy} = \frac{1}{p^2}\frac{d^2w}{dy^2} = \theta^2\frac{\partial^2 G}{\partial\eta^2}.$$

Consequently

$$2l = p\frac{d^2w}{dx^2} + 2q\frac{d^2w}{dx\,dy} - p\frac{d^2w}{dy^2} = p\theta^2\frac{\partial^2 G}{\partial\eta^2},$$

$$2m = q\theta^2\frac{\partial^2 G}{\partial\eta^2},$$

$$2n = -\theta^2\frac{\partial^2 G}{\partial\eta^2}.$$

Now we should have

$$\frac{dv}{du} = n\frac{dz}{du} = (1+\eta)\,n,$$

that is,

$$f'(u) = -\tfrac{1}{2}\theta^2(1+\eta)\frac{\partial^2 G}{\partial\eta^2},$$

from which the form of G is given by

$$G = A(u) + \eta B(u) - \frac{2}{\theta^2}f'(u)\left[(1+\eta)\{\log(1+\eta) - 1\}\right],$$

A and B being arbitrary functions.

The form of G is, however, not so important for the present purpose as is the value of $\partial^2 G/\partial\eta^2$. We deduce

$$\tfrac{1}{2}\theta^2\frac{\partial^2 G}{\partial\eta^2} = -\frac{f'(u)}{1+\eta},$$

and consequently

$$\frac{l}{p} = \frac{m}{q} = \frac{n}{-1} = \frac{f'(u)}{1 + xp' + yq'},$$

being the proper values of l, m, n, as given by

$$\left.\begin{aligned} v &= f(u)\\ z &= u + xp + yq \end{aligned}\right\}.$$

The particular solution is thus seen to be included in the general solution defined by the equations of § 31.

35. It is worth inquiring whether, in the notation of the preceding articles, there is any solution of the potential equation which is a function of u and η alone, say

$$v = F(u, \eta),$$

other than the solution $v =$ function of u alone. When the variables are taken to be x, y, u, the equation to be satisfied by v is

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} - \frac{2}{\Delta}\left(p\frac{d^2v}{dx\,du} + q\frac{d^2v}{dy\,du}\right) = 0,$$

where

$$\Delta = 1 + xp' + yq' = 1 + \eta.$$

Now

$$\left(p\frac{d}{dx} + q\frac{d}{dy}\right)\eta = pp' + qq' = 0,$$

so that, as

$$\frac{dv}{du} = \frac{\partial F}{\partial u} + (xp'' + yq'')\frac{\partial F}{\partial \eta},$$

we have

$$p\frac{d^2v}{dx\,du} + q\frac{d^2v}{dy\,du} = (pp'' + qq'')\frac{\partial F}{\partial \eta} = -(p'^2 + q'^2)\frac{\partial F}{\partial \eta} = \theta^2\frac{\partial F}{\partial \eta};$$

also

$$\frac{dv}{dx} = p'\frac{\partial F}{\partial \eta}, \qquad \frac{d^2v}{dx^2} = p'^2\frac{\partial^2 F}{\partial \eta^2},$$

$$\frac{dv}{dy} = q'\frac{\partial F}{\partial \eta}, \qquad \frac{d^2v}{dy^2} = q'^2\frac{\partial^2 F}{\partial \eta^2};$$

and therefore

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = (p'^2 + q'^2)\frac{\partial^2 F}{\partial \eta^2} = -\theta^2\frac{\partial^2 F}{\partial \eta^2}.$$

Hence the equation becomes

$$\frac{\partial^2 F}{\partial \eta^2} + \frac{2}{1 + \eta}\frac{\partial F}{\partial \eta} = 0,$$

and therefore

$$F = \frac{\psi(u)}{1 + \eta} + \phi(u),$$

where ϕ and ψ are arbitrary functions. We thus have the theorem*—

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* This result can also be expressed in the form symmetrical as regards the three variables. When thus modified, we have the theorem—

If p, q, r, be three arbitrary functions of u such that

$$p^2 + q^2 + r^2 = 0,$$

and if u be determined as a function of x, y, z, by the equation

$$au = xp + yq + zr,$$

where a is any constant, then, writing

$$v = \frac{G(u)}{a - xp' - yq' - zr'} + F(u),$$

where F and G are arbitrary functions, v satisfies LAPLACE'S *equation*

$$\nabla^2 v = 0.$$

This theorem also was stated in the paper referred to (§ 33, note).

If $p(u)$ *and* $q(u)$ *denote any functions of* u *satisfying the equation*

$$p^2 + q^2 + 1 = 0,$$

and if u *be defined by the equation*

$$z = u + xp(u) + yq(u),$$

then

$$v = \phi(u) + \frac{\psi(u)}{1 + xp'(u) + yq'(u)},$$

where ϕ *and* ψ *are arbitrary functions, satisfies* LAPLACE'S *equation*

$$\nabla^2 v = 0.$$

36. The solution just obtained can, like the solution of § 33, be connected with the general solution. Owing to the linear form of all the equations and of the expression for u, it will be sufficient, in the first instance, to take the part

$$v = \frac{\psi(u)}{1+\eta} = \frac{\psi}{1+\eta},$$

say ; because the term $\phi(u)$ in v is, in effect, identified by the preceding case. We thus must have

$$p\frac{dw}{dx} + q\frac{dw}{dy} = \frac{\psi}{1+\eta},$$

the most general solution of which is

$$w = \frac{x}{1+\eta}\frac{\psi}{p} + \mathrm{H}(u, \eta),$$

where, so far as concerns this relation, H is any arbitrary function of both its arguments. We have

$$\begin{aligned}
\frac{d^2w}{dx^2} &= -\frac{2}{(1+\eta)^2}\frac{p'\psi}{p} + \frac{2x}{(1+\eta)^3}\frac{p'^2\psi}{p} + p'^2\frac{\partial^2\mathrm{H}}{\partial\eta^2},\\
\frac{d^2w}{dx\,dy} &= -\frac{1}{(1+\eta)^2}\frac{q'\psi}{p} + \frac{2x}{(1+\eta)^3}\frac{p'q'\psi}{p} + p'q'\frac{\partial^2\mathrm{H}}{\partial\eta^2},\\
\frac{d^2w}{dy^2} &= \qquad\qquad\qquad\;\; \frac{2x}{(1+\eta)^3}\frac{q'^2\psi}{p} + q'^2\frac{\partial^2\mathrm{H}}{\partial\eta^2};
\end{aligned}$$

and therefore, after some simple reductions on substituting in the expressions of § 31, we have

$$\left.\begin{aligned}
l &= \tfrac{1}{2}\theta^2 p\frac{\partial^2\mathrm{H}}{\partial\eta^2} + \frac{x}{(1+\eta)^3}\theta^2\psi\\
m &= \tfrac{1}{2}\theta^2 q\frac{\partial^2\mathrm{H}}{\partial\eta^2} + \frac{x}{(1+\eta)^3}\theta^2\frac{q}{p}\psi - \frac{1}{(1+\eta)^2}\theta\frac{\psi}{p}\\
n &= -\tfrac{1}{2}\theta^2\frac{\partial^2\mathrm{H}}{\partial\eta^2} - \frac{x}{(1+\eta)^3}\theta^2\frac{\psi}{p} - \frac{1}{(1+\eta)^2}\theta\frac{q\psi}{p}
\end{aligned}\right\}.$$

But

$$p\frac{dw}{dx}+q\frac{dw}{dy}=\frac{\psi}{1+\eta},$$

so that

$$\frac{d}{du}\left(p\frac{dw}{dx}+q\frac{dw}{dy}\right)=\frac{\psi'}{1+\eta}-\frac{\psi}{(1+\eta)^2}(xp''+yq'');$$

and then, as

$$\frac{1}{1+\eta}\frac{d}{du}\left(p\frac{dw}{dx}+q\frac{dw}{dy}\right)=\tfrac{1}{2}\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}\right),$$

we have

$$\frac{\psi'}{(1+\eta)^2}-\frac{\psi}{(1+\eta)}(xp''+yq'')=-\tfrac{1}{2}\theta^2\frac{\partial^2\mathrm{H}}{\partial\eta^2}-\frac{x}{(1+\eta)^3}\theta^2\frac{\psi}{p}-\frac{1}{(1+\eta)^2}\theta\frac{q\psi}{p},$$

an equation to determine H.

Substituting the value of $\partial^2\mathrm{H}/\partial\eta^2$ in the above expressions for l, m, n, we find

$$\left.\begin{aligned}
l &= -\frac{1}{(1+\eta)^2}p\psi'-\frac{1}{(1+\eta)^2}p'\psi+\frac{1}{(1+\eta)^3}(xp''+yq'')p\psi\\
m &= -\frac{1}{(1+\eta)^2}q\psi'-\frac{1}{(1+\eta)^2}q'\psi+\frac{1}{(1+\eta)^3}(xp''+yq'')q\psi\\
n &= \frac{1}{(1+\eta)^2}\psi' \qquad\qquad\qquad\qquad -\frac{1}{(1+\eta)^3}(xp''+yq'')\psi
\end{aligned}\right\},$$

which are the proper values of l, m, n, as given by

$$\left.\begin{aligned}
v &= \frac{\psi(u)}{1+xp'+yq'}\\
z &= u+xp+yq
\end{aligned}\right\}.$$

The value of H is not required for the preceding investigation; but it can be actually deduced from the preceding equation. We have

$$xp''+yq''=\frac{q''}{q'}(\eta-xp')+xp''=\frac{q''}{q'}\eta+\frac{x}{q'}(p''q'-q''p'),$$

also

$$\begin{aligned}
p''q'-q''p' &= \frac{1}{q}(-p''pp'-q''qp')=\frac{-p'}{q}(pp''+qq'')\\
&= \frac{p'}{q}(p'^2+q'^2)=-\theta^2\frac{p'}{q},
\end{aligned}$$

so that

$$\frac{x}{q'}(p''q'-q''p')=-\theta^2\frac{p'}{qq'}x=\frac{\theta^2x}{p}.$$

Thus the equation for H becomes

$$\frac{\psi'}{(1+\eta)^2}-\left(\frac{q''}{q'}\eta+\theta^2\frac{x}{p}\right)\frac{\psi}{(1+\eta)^3}=-\tfrac{1}{2}\theta^2\frac{\partial^2\mathrm{H}}{\partial\eta^2}-\frac{x}{(1+\eta)^3}\frac{\theta^2\psi}{p}-\frac{1}{(1+\eta)^2}\frac{p'\psi}{p};$$

the term involving x explicitly vanishes, as it should, and therefore we have

$$\begin{aligned}\theta^2\frac{\partial^2 \mathrm{H}}{\partial\eta^2} &= -\frac{2}{(1+\eta)^2}\psi' - \frac{2}{(1+\eta)^2}\frac{p'\psi}{p} + \frac{2\eta}{(1+\eta)^3}\frac{q''}{q'}\psi \\ &= -\frac{2}{(1+\eta)^3}\frac{q''}{q'}\psi + \frac{2}{(1+\eta)^2}\left\{\left(\frac{q''}{q'} - \frac{p'}{p}\right)\psi - \psi'\right\} \\ &= -\frac{2}{(1+\eta)^3}\frac{q''}{q'}\psi + \frac{2}{(1+\eta)^2}\left\{\frac{\theta'}{\theta}\psi - \psi'\right\}.\end{aligned}$$

Hence

$$\mathrm{H}(u, \eta) = \frac{1}{1+\eta}\frac{q''}{q'\theta^2}\psi + \eta\mathrm{B}(u) + \mathrm{A}(u) + 2\frac{\theta'\psi - \psi'\theta}{\theta^3}[(1+\eta)\{\log(1+\eta) - 1\}],$$

the explicit value of H, where A and B are arbitrary functions; and thus the value of w is known. But, as already remarked, the value of $\partial^2\mathrm{H}/\partial\eta^2$ is sufficient for the identification of the solution.

37. After the preceding investigation, it is natural to inquire whether a solution more general than that which is expressible in terms of u and η, and the form of which has been suggested by the theory, can be obtained directly from the original differential equation.

When the variables are taken to be x, y, u, where

$$\left.\begin{aligned} z &= xp + yq + u \\ 0 &= p^2 + q^2 + 1 \end{aligned}\right\},$$

p and q being functions of u, the equation $a + b + c = 0$ becomes

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} - \frac{2}{1+\eta}\left(p\frac{d^2v}{dx\,du} + q\frac{d^2v}{dy\,du}\right) = 0,$$

where

$$\eta = xp' + yq'.$$

Let ζ be defined by the equation

$$\zeta = xp'' + yq'',$$

so that u, η, ζ, are three variables functionally equivalent to x, y, z; and let us inquire what solutions of the form

$$v = \mathrm{F}(u, \eta, \zeta)$$

are possessed by the foregoing equation. We have

$$\begin{aligned} p' &= q\theta, & q' &= -p\theta, \\ p'' &= q\theta' - p\theta^2, & q'' &= -p\theta' - q\theta^2, \\ p''' &= q\theta'' - 3p\theta\theta' - q\theta^3, & q''' &= -p\theta'' - 3q\theta\theta' + p\theta^3; \end{aligned}$$

so that

$$\begin{aligned} p'^2 + q'^2 &= -\theta^2, \\ p'p'' + q'q'' &= -\theta\theta', \\ p''^2 + q''^2 &= -\theta'^2 - \theta^4, \\ pp'' + qq'' &= \theta^2, \\ pp''' + qq''' &= 3\theta\theta', \end{aligned}$$

quantities which occur in the substitution. Now

$$\frac{dv}{dx} = p' \frac{\partial F}{\partial \eta} + p'' \frac{\partial F}{\partial \zeta},$$

$$\frac{d^2v}{dx^2} = p'^2 \frac{\partial^2 F}{\partial \eta^2} + 2p'p'' \frac{\partial^2 F}{\partial \eta \, \partial \zeta} + p''^2 \frac{\partial^2 F}{\partial \zeta^2};$$

similarly

$$\frac{d^2v}{dy^2} = q'^2 \frac{\partial^2 F}{\partial \eta^2} + 2q'q'' \frac{\partial^2 F}{\partial \eta \, \partial \zeta} + q''^2 \frac{\partial^2 F}{\partial \zeta^2};$$

and therefore

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = -\theta^2 \frac{\partial^2 F}{\partial \eta^2} - 2\theta\theta' \frac{\partial^2 F}{\partial \eta \, \partial \zeta} - (\theta'^2 + \theta^4) \frac{\partial^2 F}{\partial \zeta^2}.$$

Again,

$$\left(p \frac{d}{dx} + q \frac{d}{dy}\right) \eta = 0,$$

$$\left(p \frac{d}{dx} + q \frac{d}{dy}\right) \zeta = pp'' + qq'' = \theta^2,$$

so that, as

$$\frac{dv}{du} = \frac{\partial F}{\partial u} + \zeta \frac{\partial F}{\partial \eta} + (xp''' + yq''') \frac{\partial F}{\partial \zeta},$$

we have

$$\begin{aligned} p \frac{d^2v}{dx\,du} + q \frac{d^2v}{dy\,du} = \theta^2 \frac{\partial^2 F}{\partial u \, \partial \zeta} + \theta^2 \frac{\partial F}{\partial \eta} + \theta^2 \zeta \frac{\partial^2 F}{\partial \eta \, \partial \zeta} \\ + (pp''' + qq''') \frac{\partial F}{\partial \zeta} + \theta^2 (xp''' + yq''') \frac{\partial^2 F}{\partial \zeta^2}. \end{aligned}$$

Taking account of the values of η and ζ, it is not difficult to prove that

$$\theta^2 (xp''' + yq''') = \eta (\theta\theta'' - 3\theta'^2 - \theta^2) + \zeta 3\theta\theta';$$

consequently

$$\begin{aligned} p \frac{d^2v}{dx\,du} + q \frac{d^2v}{dy\,du} = \theta^2 \left\{ \frac{\partial^2 F}{\partial u \, \partial \zeta} + \zeta \frac{\partial^2 F}{\partial \eta \, \partial \zeta} + \frac{\partial F}{\partial \eta} \right\} \\ + 3\theta\theta' \frac{\partial F}{\partial \zeta} + \{\eta (\theta\theta'' - 3\theta'^2 - \theta^4) + 3\theta\theta'\zeta\} \frac{\partial^2 F}{\partial \zeta^2} \end{aligned}$$

and the equation satisfied by F becomes

$$\begin{aligned} \theta^2 \frac{\partial^2 F}{\partial \eta^2} + 2 \frac{\theta^2}{1 + \eta} \left(\frac{\partial F}{\partial \eta} + \frac{\partial^2 F}{\partial u \, \partial \zeta} \right) + 2 \left(\theta\theta' + \frac{\theta^2}{1 + \eta} \zeta \right) \frac{\partial^2 F}{\partial \eta \, \partial \zeta} + 6 \frac{\theta\theta'}{1 + \eta} \frac{\partial F}{\partial \zeta} \\ + \frac{1}{1 + \eta} \{\theta'^2 + \theta^4 + \eta (2\theta\theta'' - 5\theta'^2 - \theta'') + 6\theta\theta'\zeta\} \frac{\partial^2 F}{\partial \zeta^2} = 0. \end{aligned}$$

We can at once infer the result of § 35 as to the solutions which, being of the suggested form, explicitly involve u and η only; but other solutions in a finite form do not suggest themselves.

38. It need only be remarked that, with the slightest change of notation, all the preceding results can be associated also with the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = c^2 \frac{\partial^2 v}{\partial t^2},$$

where c is a constant.

Application to $\nabla^2 v + \kappa^2 v = 0$.

39. Consider next, but more briefly, the equation

$$\nabla^2 v + \kappa^2 v = 0,$$

where κ is a constant; in our notation, it may be written

$$a + b + c + \kappa^2 v = 0.$$

The characteristic invariant equation is

$$\mathrm{I} = p^2 + q^2 + 1 = 0,$$

the same as for $\nabla^2 v = 0$. Substituting for a and b their values as given in § 1, it is seen that, on account of the characteristic equation, the term in c disappears from the result; and we have

$$\mathrm{J} = \frac{dl}{dx} - p\frac{dn}{dx} + \frac{dm}{dy} - q\frac{dn}{dy} + \kappa^2 v = 0.$$

The subsidiary system thus is

$$\left.\begin{array}{l} \mathrm{I} = 0, \quad \mathrm{J} = 0 \\ \dfrac{dm}{dx} - p\dfrac{dn}{dy} = \dfrac{dl}{dy} - q\dfrac{dn}{dx} \\ \dfrac{dv}{dx} = l + np, \quad \dfrac{dv}{dy} = m + nq \end{array}\right\},$$

the third of which is an analytical consequence of the fourth and the fifth. As in the preceding example, the general solution of $\mathrm{I} = 0$ is

$$z = u + xp(u) + yq(u),$$

where p and q are functions subject to the relation

$$\{p(u)\}^2 + \{q(u)\}^2 + 1 = 0;$$

and in the subsidiary system the differentiations with regard to x and y are partial on the supposition that u is the third variable: that is, in these partial differentiations in the subsidiary system, u does not vary. Consequently $J = 0$ is expressible in the form

$$\frac{d}{dx}(l - np) + \frac{d}{dy}(m - nq) + \kappa^2 v = 0\,;$$

and therefore

$$\left.\begin{aligned} \frac{d^2}{dx^2}(l - np) + \frac{d^2}{dx\,dy}(m - nq) &= -\kappa^2\frac{dv}{dx} = -\kappa^2(l + np) \\ \frac{d^2}{dx\,dy}(l - np) + \frac{d^2}{dy^2}(m - nq) &= -\kappa^2\frac{dv}{dy} = -\kappa^2(m + nq) \end{aligned}\right\},$$

so that

$$\left.\begin{aligned} \left(\frac{d^2}{dx^2} + \kappa^2\right) l + \frac{d^2}{dx\,dy} m &= \left(p\frac{d^2}{dx^2} + q\frac{d^2}{dx\,dy} - p\kappa^2\right) n \\ \frac{d^2}{dx\,dy} l + \left(\frac{d^2}{dy^2} + \kappa^2\right) m &= \left(p\frac{d^2}{dx\,dy} + q\frac{d^2}{dy^2} - q\kappa^2\right) n \end{aligned}\right\}.$$

Operate on the first equation with $d^2/dy^2 + \kappa^2$, on the second with $d^2/dx\,dy$, and subtract; then, dividing by κ^2, we have

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2\right) l = \left(p\frac{d^2}{dx^2} + 2q\frac{d^2}{dx\,dy} - p\frac{d^2}{dy^2} - p\kappa^2\right) n.$$

Similarly, operating on the first equation with $d^2/dx\,dy$, on the second with $d^2/dx^2 + \kappa^2$, subtracting, and dividing by κ^2, we have

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2\right) m = \left(-q\frac{d^2}{dx^2} + 2p\frac{d^2}{dx\,dy} + q\frac{d^2}{dy^2} - q\kappa^2\right) m.$$

It therefore follows that some function w of x, y, and u exists such that

$$\left.\begin{aligned} l &= p\frac{d^2w}{dx^2} + 2q\frac{d^2w}{dx\,dy} - p\frac{d^2w}{dy^2} - p\kappa^2 w \\ m &= -q\frac{d^2w}{dx^2} + 2p\frac{d^2w}{dx\,dy} + q\frac{d^2w}{dy^2} - q\kappa^2 w \\ n &= \frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \kappa^2 w \end{aligned}\right\};$$

but, so far as the subsidiary system is concerned, w is perfectly arbitrary—a result in accordance with the general explanation of § 30, there being only four functionally independent equations for the preliminary determination of five quantities. Also

$$l + np = 2\left(p\frac{d^2w}{dx^2} + q\frac{d^2w}{dx\,dy}\right) = \frac{dv}{dx},$$

$$m + nq = 2\left(p\frac{d^2w}{dx\,dy} + q\frac{d^2w}{dy^2}\right) = \frac{dv}{dy};$$

hence

$$v = 2\left(p\frac{dw}{dx} + q\frac{dw}{dy}\right),$$

no arbitrary function of u needing to be added, for it can be considered as accounted for in the arbitrary function w.

But now, regarding variations of u, we must have the equation

$$\frac{dv}{du} = n\frac{dz}{du}$$

satisfied; that is, the function w must satisfy the equation

$$2\frac{d}{du}\left(p\frac{dw}{dx} + q\frac{dw}{dy}\right) = \Delta\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \kappa^2 w\right),$$

where Δ denotes $1 + xp' + yq'$. As in the former example, this equation imposes the limitation upon the arbitrariness of w; when its general integral is known, the most general value of v can be deduced.

40. It is an inference from the general theory that the quantity u, determined as a function of x, y, z, by the equations

$$\left.\begin{aligned} 0 &= 1 + \{p(u)\}^2 + \{q(u)\}^2 \\ z &= u + xp(u) + yq(u) \end{aligned}\right\},$$

is an argument of the arbitrary functions that occur in the solution of the equation $\nabla^2 v + \kappa^2 v = 0$. We therefore transform the variables from x, y, z, to x, y, u, where u now is known; and the result, obtained by analysis similar to that in § 33, is

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \kappa^2 v - \frac{2}{\Delta}\left(p\frac{d^2v}{dx\,du} + q\frac{d^2v}{dy\,du}\right) = 0.$$

Manifestly a function of u alone is not a solution of this equation; but, on the analogy of § 35, we are led to consider what solutions (if any) of the form

$$v = F(u, \eta)$$

are possessed by the equation, η having its former value $xp' + yq'$. When this value is substituted, the equation takes the form

$$-\theta^2\frac{\partial^2 F}{\partial\eta^2} - 2\frac{\theta^2}{1+\eta}\frac{\partial F}{\partial\eta} + \kappa^2 F = 0,$$

where

$$-\theta^2 = p'^2 + q'^2.$$

Hence

$$\frac{\partial^2}{\partial\eta^2}\{(1+\eta)F\} = \frac{\kappa^2}{\theta^2}(1+\eta)F,$$

so that, as θ is a function of u only, we have

$$(1+\eta)\,\mathrm{F} = \mathrm{A}e^{\frac{\kappa}{\theta}\eta} + \mathrm{B}e^{-\frac{\kappa}{\theta}\eta},$$

where A and B are independent of η, that is, are arbitrary functions of u. We thus have the theorem*—

If $p(u)$ *and* $q(u)$ *denote any functions of* u *satisfying the equation*

$$p^2 + q^2 + 1 = 0,$$

and if u *be defined as a function of* x, y, z, *by the equation*

$$z = u + xp(u) + yq(u),$$

also if v *denote*

$$\frac{\phi(u)\,e^{i\kappa(xp+yq')(p'^2+q'^2)^{-\frac{1}{2}}} + \psi(u)\,e^{-i\kappa(xp'+yq')(p'^2+q'^2)^{-\frac{1}{2}}}}{1 + xp' + yq'},$$

ϕ *and* ψ *being arbitrary functions of* u, *then* v *satisfies the equation*

$$\nabla^2 v + \kappa^2 v = 0.$$

The connection between this solution and the general solution indicated in § 39 can be established as in the corresponding case of the potential equation (§§ 34, 36).

Application to $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = \mu\dfrac{\partial v}{\partial t}$.

41. The preceding examples have led, in each case, to a solution which (though not the most general) was expressible in a finite form. We now take one other example, viz.,

$$\gamma l + b + c = 0,$$

which can be regarded as the equation for the variable conduction of heat in two

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* This result can also be expressed (see the paper already quoted, § 33, note) in a form symmetrical as regards the three variables, as follows:—

If p, q, r, *be three arbitrary functions of* u *such that*

$$\{p(u)\}^2 + \{q(u)\}^2 + \{r(u)\}^2 = 0,$$

and if u *be determined as a function of* x, y, z, *by the equation*

$$au = xp(u) + yq(u) + zr(u),$$

a *being any constant, also if* v *denote*

$$\frac{\phi(u)\,e^{i\kappa(xp'+yq'+zr')(p'^2+q'^2+r'^2)^{-\frac{1}{2}}} + \psi(u)\,e^{-i\kappa(xp'+yq'+zr')(p'^2+q'^2+r'^2)^{-\frac{1}{2}}}}{a - xp' - yq' - zr'},$$

where ϕ *and* ψ *denote arbitrary functions of* u, *then* v *satisfies the equation*

$$\nabla^2 v + \kappa^2 v = 0.$$

dimensions; it will be seen that the general solution is not expressible in a finite form.

The characteristic equation is

$$q^2 + 1 = 0,$$

say $q = -i$; and then u is determined by

$$z + iy = xp(u) + u,$$

where $p(u)$ is arbitrary. The subsidiary system is

$$\left.\begin{array}{c} \gamma l + \dfrac{dm}{dy} + i\dfrac{dn}{dy} = 0 \\ \dfrac{dv}{dx} = l + np, \qquad \dfrac{dv}{dy} = m - ni \end{array}\right\}.$$

We easily find

$$\left.\begin{array}{r} \left(\dfrac{d^2}{dy^2} + \gamma\dfrac{d}{dx}\right)v + \left(2i\dfrac{d}{dy} - \gamma p\right)n = 0 \\ \left(\dfrac{d^2}{dy^2} + \gamma\dfrac{d}{dx}\right)m + \left(i\dfrac{d^2}{dy^2} - \gamma p\dfrac{d}{dy} - i\gamma\dfrac{d}{dx}\right)n = 0 \end{array}\right\};$$

and so we have

$$\left.\begin{array}{l} v = 2i\dfrac{dw}{dy} - \gamma pw \\ n = -\dfrac{d^2w}{dy^2} - \gamma\dfrac{dw}{dx} \\ m = i\dfrac{d^2w}{dy^2} - \gamma p\dfrac{dw}{dy} - i\gamma\dfrac{dw}{dx} \\ l = 2i\dfrac{d^2w}{dx\,dy} + p\dfrac{d^2w}{dy^2} \end{array}\right\},$$

where w is an arbitrary function of x, y, and u. But we also must have the equation

$$\frac{dv}{du} = n\frac{dz}{du}$$

satisfied; that is, we must satisfy the equation

$$2i\frac{d^2w}{dy\,du} - \gamma p\frac{dw}{du} - \gamma wp' = -i\{1 + xp'(u)\}\left\{\frac{d^2w}{dy^2} + \gamma\frac{dw}{dx}\right\}.$$

If a general value of w can be deduced, then the general value of v could be constructed, and conversely.

42. Taking $z + iy = s$, $z - iy = s'$, the equation can be written in the form

$$\frac{\partial^2 v}{\partial s\,\partial s'} + \tfrac{1}{4}\gamma\frac{\partial v}{\partial x} = 0,$$

a solution of which is given by

$$v = \mathrm{F}(s, s') - 4\frac{x}{\gamma}\frac{\partial^2 \mathrm{F}}{\partial s\, \partial s'} + \frac{16x^2}{\gamma^2}\frac{1}{2!}\frac{\partial^4 \mathrm{F}}{\partial s^2\, \partial s'^2} - \ldots,$$

where F is any arbitrary function of s and s'. Now the solution, which appears to be most frequently useful, of the original equation is obtained by taking

$$v = \mathrm{V}e^{-\mu x},$$

where μ is constant, and V is independent of x. In this case,

$$\frac{\partial v}{\partial x} = -\mu \mathrm{V} e^{-\mu x},$$

and the equation for V is

$$\frac{\partial^2 \mathrm{V}}{\partial s\, \partial s'} = \tfrac{1}{4}\gamma\mu \mathrm{V}.$$

To identify the two forms, we must have

$$\begin{aligned} \mathrm{F}(s, s') - x\frac{4}{\gamma}\frac{\partial^2 \mathrm{F}}{\partial s\, \partial s'} + \frac{x^2}{2!}\left(\frac{4}{\gamma}\right)^2 \frac{\partial^4 \mathrm{F}}{\partial s^2\, \partial s'^2} - \ldots, \\ = \mathrm{V}e^{-\mu x} \\ = \mathrm{V} - x\mu\,\mathrm{V} + \frac{x^2}{2!}\mu^2 \mathrm{V} - \ldots; \end{aligned}$$

consequently

$$\left.\begin{aligned} \mathrm{V} &= \mathrm{F}(s, s') \\ \mu^n \mathrm{V} &= \left(\frac{4}{\gamma}\right)^n \frac{\partial^{2n} \mathrm{F}}{\partial s^n\, \partial s'^n} \end{aligned}\right\}$$

for all values of n. All the conditions are satisfied in virtue of

$$\frac{\partial^2 \mathrm{V}}{\partial s\, \partial s'} = \tfrac{1}{4}\gamma\mu \mathrm{V};$$

and consequently the more general solution above obtained includes the less general solution customarily used when the arbitrariness of F is made subject to the equation

$$\frac{\partial^2 \mathrm{F}}{\partial s\, \partial s'} = \tfrac{1}{4}\gamma\mu \mathrm{F}.$$

This equation is of LAPLACE's linear form with equal invariants, which, moreover, are constants; hence the whole series* of derived invariants is unlimited in number, and the number of derivatives of an arbitrary function of s and the number of derivatives of an arbitrary function of s', that occur in the most general solution, are unlimited in each case: that is, the solution is not expressible in finite terms.

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* DARBOUX, 'Théorie Générale des Surfaces,' t. ii.

43. To see how this solution is included in the general solution of § 41, it is necessary to determine a function w such that

$$2i\frac{dw}{dy} - \gamma p w = v = e^{-4\frac{x}{\gamma}\frac{\delta^2}{\delta s\,\delta s'}}\,\mathrm{F}(s, s') = \Phi(s, s'),$$

say. Now, for the derivatives with regard to y, s is parametric, being $xp + u$; and $s' = s - 2iy$. We thus find on integration

$$e^{\frac{1}{2}i\gamma p y}\,w = \phi(x, u) + \frac{1}{2i}\int e^{\frac{1}{2}i\gamma p y}\,\Phi(s, s - 2iy)\,dy,$$

ϕ being arbitrary, and s parametric in the integral. Evaluating the integral through integration by parts, and dividing by the exponential factor on the left-hand side, we find

$$\begin{aligned} w &= e^{-\frac{1}{2}i\gamma p y}\phi(x, u) - \frac{1}{\gamma p}\left\{\Phi(s, s') + \sum_{n=1}^{\infty}\left(\frac{4}{\gamma p}\right)^n \frac{d^n}{ds'^n}\Phi(s, s')\right\} \\ &= e^{-\frac{1}{2}i\gamma p y}\phi(x, u) - \frac{1}{\gamma p}\sum_{n=0}^{\infty}\left(\frac{4}{\gamma p}\right)^n \frac{d^n\Phi}{ds'^n} \\ &= w_1 - w_2, \end{aligned}$$

say, so that w_1 is expressed in terms of x, y, u, and w_2 in terms of s, s', x.

Denoting derivatives of w_2 with regard to s, s', x, by $\delta/\delta s$, $\delta/\delta s'$, $\delta/\delta x$, respectively, we have

$$\frac{d}{dx} = p\left(\frac{\delta}{\delta s} + \frac{\delta}{\delta s'}\right) + \frac{\delta}{\delta x} = p\left(\frac{\delta}{\delta s} + \frac{\delta}{\delta s'}\right) - \frac{4}{\gamma}\frac{\delta^2}{\delta s\,\delta s'},$$

this last change arising from the fact that every term in w_2 satisfies the transformed equation because Φ satisfies it;

$$\frac{d}{dy} = -2i\frac{\delta}{\delta s'}, \qquad \frac{d}{du} = (1 + xp')\left(\frac{\delta}{\delta s} + \frac{\delta}{\delta s'}\right).$$

Consequently

$$\begin{aligned} \frac{dw_2}{dx} &= p\frac{\delta w_2}{\delta s} + p\frac{\delta w_2}{\delta s'} - \frac{4}{\gamma}\frac{\delta^2 w_2}{\delta s\,\delta s'}, \\ \frac{dw_2}{dy} &= -2i\frac{\delta w_2}{\delta s'}, \\ \frac{d^2 w_2}{dy^2} &= -4\frac{\delta^2 w_2}{\delta s'^2}, \\ \frac{d^2 w_2}{dx\,dy} &= -2ip\frac{\delta^2 w_2}{\delta s\,\delta s'} - 2ip\frac{\delta^2 w_2}{\delta s'^2} + \frac{8i}{\gamma}\frac{\delta^3 w_2}{\delta s\,\delta s'^2}. \end{aligned}$$

Further,

$$\frac{d^2 w_1}{dy^2} + \gamma\frac{dw_1}{dx} = e^{-\frac{1}{2}i\gamma p y}\left(\gamma\frac{d\phi}{dx} - \tfrac{1}{4}\gamma^2 p^2\phi\right),$$

so that

$$\begin{aligned} -n &= \frac{d^2w}{dy^2} + \gamma\frac{dw}{dx} \\ &= e^{-\frac{1}{2}i\gamma py}\left(\gamma\frac{d\phi}{dx} - \tfrac{1}{4}\gamma^2p^2\phi\right) + 4\frac{\delta^2w_2}{\delta s'^2} - \gamma p\frac{\delta w_2}{\delta s} - \gamma p\frac{\delta w_2}{\delta s'} + 4\frac{\delta^2 w_2}{\delta s\,\delta s'} \\ &= e^{-\frac{1}{2}i\gamma py}\left(\gamma\frac{d\phi}{dx} - \tfrac{1}{4}\gamma^2p^2\phi\right) - \left(\frac{\delta}{\delta s} + \frac{\delta}{\delta s'}\right)\left(\gamma p w_2 - 4\frac{\delta w_2}{\delta s'}\right). \end{aligned}$$

But

$$w_2 - \frac{4}{\gamma p}\frac{\delta w_2}{\delta s'} = \frac{1}{\gamma p}\Phi;$$

and therefore

$$-n = e^{-\frac{1}{2}i\gamma py}\left(\gamma\frac{d\phi}{dx} - \tfrac{1}{4}\gamma^2p^2\phi\right) - \left(\frac{\delta}{\delta s} + \frac{\delta}{\delta s'}\right)\Phi.$$

Now we must have

$$\frac{dv}{du} = n\frac{dz}{du},$$

that is,

$$(1 + xp')\left(\frac{\delta\Phi}{\delta s} + \frac{\delta\Phi}{\delta s'}\right) = (1 + xp')\,n,$$

and so

$$n = \left(\frac{\delta}{\delta s} + \frac{\delta}{\delta s'}\right)\Phi.$$

But, because $v = \Phi$, $s = z + iy$, $s' = z - iy$, we have

$$\frac{\partial v}{\partial z} = \left(\frac{\delta}{\delta s} + \frac{\delta}{\delta s'}\right)\Phi = n;$$

that is, the foregoing value of n is the z-derivative of v. Further, comparing the two values of n, we find

$$\gamma\frac{d\phi}{dx} - \tfrac{1}{4}\gamma^2p^2\phi = 0,$$

whence

$$\phi = e^{\frac{1}{4}\gamma p^2x}\,\mathrm{A}(u),$$

A being an arbitrary function; hence

$$w = e^{\frac{1}{4}\gamma p(px-2iy)}\,\mathrm{A}(u) - \frac{1}{\gamma p}\sum_{n=0}^{\infty}\left(\frac{4}{\gamma p}\right)^n\frac{d^n\Phi}{ds'^n},$$

the explicit value of w.

Next, for the value of l, we have

$$p\frac{d^2w_1}{dy^2} + 2i\frac{d^2w_1}{dx\,dy} = \frac{d}{dy}\left(p\frac{dw_1}{dy} + 2i\frac{dw_1}{dx}\right) = 0$$

from the value of w, just obtained; and

$$p\frac{d^2w_2}{dy^2} + 2i\frac{d^2w_2}{dx\,dy} = -4p\frac{\delta^2w_2}{\delta s'^2} + 4p\frac{\delta^2w_2}{\delta s\,\delta s'} + 4p\frac{\delta^2w_2}{\delta s'^2} - \frac{16}{\gamma}\frac{\delta^3w_2}{\delta s\,\delta s'^2}$$

$$= 4\frac{\delta^2}{\delta s\,\delta s'}\left(p - \frac{4}{\gamma}\frac{\delta}{\delta s'}\right)w_2$$

$$= 4\frac{\delta^2}{\delta s\,\delta s'}\frac{\Phi}{\gamma}$$

$$= \frac{4}{\gamma}\frac{\delta^2\Phi}{\delta s\,\delta s'}.$$

Hence

$$l = p\frac{d^2w}{dy^2} + 2i\frac{d^2w}{dx\,dy} = -\frac{4}{\gamma}\frac{\delta^2\Phi}{\delta s\,\delta s'} = \frac{\partial\Phi}{\partial x} = \frac{\partial v}{\partial x},$$

or the value of l is the x-derivative of v.

Similarly for the value of m. The solution is thus seen to be included in the form given in § 41; moreover, the value of w obtained in the preceding investigation is a solution of the equation there required to be satisfied.

44. It has been seen (§ 27) that the equations to which the method involving derivatives of v, l, m, n, alone can be applied—which method has been indicated as the generalisation of AMPÈRE's for the case of two independent variables—belong to a distinctly limited class. Moreover, even for equations in this class, it may happen that an integrable combination of the subsidiary system cannot be constructed, or that the quasi-general process sketched in § 28 is not effective. Consequently it becomes necessary to have some other method; and this is provided by what has been indicated as the generalisation of DARBOUX's method for the case of two independent variables.

In the discussion, we shall consider only the case when it proves necessary to construct the first derivatives of an equation $F = 0$; but the explanations apply, *mutatis mutandis,* to other cases when second derivatives of $F = 0$ and derivatives of higher orders should be constructed. Examples of the latter cases, when the characteristic invariant is resoluble, have already been given in Section II.

45. The original equation is

$$F(a, b, c, f, g, h, l, m, n, v, x, y, z) = 0.$$

The characteristic invariant is

$$Ap^2 + Hpq + Bq^2 - Gp - Fq + C = 0,$$

and it is supposed to be irreducible. The subsidiary equations are

$$\left.\begin{aligned}
X + A\left(\frac{da}{dx} - p\frac{dg}{dx}\right) + H\left(\frac{dh}{dx} - p\frac{dg}{dy}\right) + B\left(\frac{dh}{dy} - q\frac{dg}{dy}\right) + G\frac{dg}{dx} + F\frac{dg}{dy} &= 0\\
Y + A\left(\frac{dh}{dx} - p\frac{df}{dx}\right) + H\left(\frac{db}{dx} - p\frac{df}{dy}\right) + B\left(\frac{db}{dy} - q\frac{df}{dy}\right) + G\frac{df}{dx} + F\frac{df}{dy} &= 0\\
Z + A\left(\frac{dg}{dx} - p\frac{dc}{dx}\right) + H\left(\frac{df}{dx} - p\frac{dc}{dy}\right) + B\left(\frac{df}{dy} - q\frac{dc}{dy}\right) + G\frac{dc}{dx} + F\frac{dc}{dy} &= 0
\end{aligned}\right\};$$

and equations of definition and derivation are

$$\left.\begin{array}{ll} \dfrac{dv}{dx} = l + np, & \dfrac{dv}{dy} = m + nq \\ \dfrac{dl}{dx} = a + qp, & \dfrac{dl}{dy} = h + gq \\ \dfrac{dm}{dx} = h + fp, & \dfrac{dm}{dy} = b + fq \\ \dfrac{dn}{dx} = g + cp, & \dfrac{dn}{dy} = f + cq \end{array}\right\}, \quad \left.\begin{array}{l} \dfrac{dv}{du} = n\dfrac{dz}{du} \\ \dfrac{dl}{du} = g\dfrac{dz}{du} \\ \dfrac{dm}{du} = f\dfrac{dz}{du} \\ \dfrac{dn}{du} = c\dfrac{dz}{du} \end{array}\right\}.$$

So far as concerns derivatives with regard to x and y, there appear to be twelve equations, viz., the characteristic invariant, the three subsidiary equations deduced by taking

$$\frac{\mathrm{DF}}{\mathrm{D}x} = 0, \qquad \frac{\mathrm{DF}}{\mathrm{D}y} = 0, \qquad \frac{\mathrm{DF}}{\mathrm{D}z} = 0,$$

and the eight equations of definition and derivation; and the number of quantities to be obtained from the subsidiary system is eleven, viz., $a, b, c, f, g, h, l, m, n, v, z$. On the other hand, $\mathrm{F} = 0$ is an integral of the system, and it is a persistent relation; consequently one of the set of three equations can be regarded as depending on the other two. Again, from the equations

$$\frac{dv}{dx} = l + np, \qquad \frac{dv}{dy} = m + nq,$$

we have

$$\frac{dl}{dy} + p\frac{dn}{dy} = \frac{dm}{dx} + q\frac{dn}{dx},$$

identically satisfied in virtue of the values of dl/dy, dm/dx, dn/dx, dn/dy, given by the other equations of the set of eight; hence one of these can be regarded as dependent functionally on the rest, and the set of eight are therefore equivalent to seven only.

The whole set of equations in the subsidiary system, involving derivatives with regard to x and y, thus contains ten independent equations; and eleven quantities are to be obtained. Hence it may be possible to express ten of these in terms of one of them, or to express the whole eleven in terms of a single new quantity, arbitrary so far as the set is concerned; its arbitrariness will then be limited so that it shall satisfy the subsidiary equations involving derivatives with regard to u.

46. *The relations thus obtained,* as satisfying all the subsidiary equations, *constitute an integral of the equation.*

This result is established by an argument similar to that in § 29 for the case when the subsidiary system is simpler; accordingly here it will not be repeated.

Application to $\nabla^2 v = 0$.

47. The investigations as to the potential equation, given in § 31, did not require the consideration of derivatives of a, b, c, f, g, h; but the example can be used to indicate the method of proceeding when such derivatives must be taken into account, as in the preceding theory.

For the equation

$$a + b + c = 0,$$

the special subsidiary equations involving derivatives of a, b, c, with regard to x and y, are

$$\left.\begin{aligned} \frac{da}{dx} - p\frac{dg}{dx} + \frac{dh}{dy} - q\frac{dg}{dy} &= 0 \\ \frac{dh}{dx} - p\frac{df}{dx} + \frac{db}{dy} - q\frac{df}{dy} &= 0 \\ \frac{dg}{dx} - p\frac{dc}{dx} + \frac{df}{dy} - q\frac{dc}{dy} &= 0 \end{aligned}\right\};$$

the characteristic equation is

$$p^2 + q^2 + 1 = 0;$$

and the relations of identity are

$$\left.\begin{aligned} \frac{da}{dy} + p\frac{dg}{dy} - \frac{dh}{dx} - q\frac{dg}{dx} &= 0 \\ \frac{dh}{dy} + p\frac{df}{dy} - \frac{db}{dx} - q\frac{df}{dx} &= 0 \\ \frac{dg}{dy} + p\frac{df}{dy} - \frac{df}{dx} - q\frac{dc}{dx} &= 0 \end{aligned}\right\}.$$

But the latter can be replaced by the equations of definition, viz.,

$$\left.\begin{aligned} \frac{dl}{dx} &= a + gp, & \frac{dl}{dy} &= h + gq, & \frac{dl}{du} &= g\frac{dz}{du} \\ \frac{dm}{dx} &= h + fp, & \frac{dm}{dy} &= b + fq, & \frac{dm}{du} &= f\frac{dz}{du} \\ \frac{dn}{dx} &= g + cp, & \frac{dn}{dy} &= f + cq, & \frac{dn}{du} &= c\frac{dz}{du} \end{aligned}\right\},$$

together with

$$\frac{dv}{dx} = l + np, \qquad \frac{dv}{dy} = m + nq, \qquad \frac{dv}{du} = n\frac{dz}{du};$$

it is, indeed, from the first two columns of these equations of definition that the relations of identity are deduced.

In the subsidiary equations and in the relations of identity, the derivatives with regard to x and y are framed on the supposition that u is constant. Now the complete solution of

$$p^2 + q^2 + 1 = 0$$

is given, by CHARPIT'S process, in the form

$$p = \text{constant}, \qquad q = \text{constant}, \qquad z - px - qy = \text{constant}.$$

But these constants arise when u is constant; hence we may take

$$z - px - qy = u,$$

where $p = p(u)$ and $q = q(u)$ are arbitrary functions of u subject to the condition

$$p^2 + q^2 + 1 = 0.$$

From the first of the three subsidiary equations, remembering that p and q are now constant so far as concerns derivatives with regard to x and y, we have

$$\frac{d}{dx}(a - pg) + \frac{d}{dy}(h - qg) = 0,$$

so that some function w of x, y, u, exists such that

$$a - pg = \frac{dw}{dy}, \qquad h - qg = -\frac{dw}{dx}.$$

Also we have

$$a + pg = \frac{dl}{dx}, \qquad h + qg = \frac{dl}{dy},$$

so that

$$2a = \frac{dw}{dy} + \frac{dl}{dx},$$

$$2h = -\frac{dw}{dx} + \frac{dl}{dy},$$

$$\left.\begin{aligned} 2pg &= -\frac{dw}{dy} + \frac{dl}{dx} \\ 2qg &= \frac{dw}{dx} + \frac{dl}{dy} \end{aligned}\right\}.$$

From the last we have

$$p\left(\frac{dw}{dx} + \frac{dl}{dy}\right) = q\left(\frac{dw}{dy} + \frac{dl}{dx}\right),$$

that is,

$$\frac{d}{dx}(pw - ql) + \frac{d}{dy}(pl + qw) = 0,$$

so that some function θ of x, y, u, exists such that

$$pw - ql = \frac{d\theta}{dy}, \qquad qw + pl = -\frac{d\theta}{dx},$$

or

$$w = -p\frac{d\theta}{dy} + q\frac{d\theta}{dx}, \qquad l = q\frac{d\theta}{dy} + p\frac{d\theta}{dx}.$$

Substituting these values above, we find

$$\left.\begin{aligned}
2a &= \left(p,\ q,\ -p\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)^2\theta\\
2h &= \left(-q,\ p,\ q\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)^2\theta\\
2g &= \left(1,\ 0,\ 1\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)^2\theta\\
l &= \left(p,\ q\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)\theta
\end{aligned}\right\},$$

θ denoting an arbitrary function of x, y, u.

The second of the three subsidiary equations similarly gives

$$\left.\begin{aligned}
2h &= \left(p,\ q,\ -p\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)^2\chi\\
2b &= \left(-q,\ p,\ q\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)^2\chi\\
2f &= \left(1,\ 0,\ 1\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)^2\chi\\
m &= \left(p,\ q\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)\chi
\end{aligned}\right\},$$

χ denoting an arbitrary function of x, y, u; and the third of the subsidiary equations gives

$$\left.\begin{aligned}
2g &= \left(p,\ q,\ -p\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)^2\psi\\
2f &= \left(-q,\ p,\ q\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)^2\psi\\
2c &= \left(1,\ 0,\ 1\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)^2\psi\\
n &= \left(p,\ q\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)\psi
\end{aligned}\right\},$$

ψ denoting an arbitrary function.

These three arbitrary functions are not independent of one another. Comparing the two values of $2f$, $2g$, $2h$, respectively which have been obtained, we see that some function ϕ of x, y, u, exists such that

$$\left.\begin{aligned}
\theta &= \left(p,\, q,\, -p \,\big)\!\!\big(\, \frac{d}{dx},\ \frac{d}{dy}\right)^2 \phi \\
\chi &= \left(-q,\, p,\, q \,\big)\!\!\big(\, \frac{d}{dx},\ \frac{d}{dy}\right)^2 \phi \\
\psi &= \left(1,\, 0,\, 1 \,\big)\!\!\big(\, \frac{d}{dx},\ \frac{d}{dy}\right)^2 \phi
\end{aligned}\right\} ;$$

manifestly ϕ is an arbitrary function of x, y, u.

To deduce the value of v so far as concerns variations of x and y, we have

$$\begin{aligned}
\frac{dv}{dx} &= l + np \\
&= \left(p,\, q \,\big)\!\!\big(\, \frac{d}{dx},\ \frac{d}{dy}\right)(\theta + p\psi) \\
&= 2\left(p\frac{d}{dx} + q\frac{d}{dy}\right)\left(p\frac{d^2\phi}{dx^2} + q\frac{d^2\phi}{dx\,dy}\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{dv}{dy} &= m + nq \\
&= \left(p,\, q \,\big)\!\!\big(\, \frac{d}{dx},\ \frac{d}{dy}\right)(\chi + q\psi) \\
&= 2\left(p\frac{d}{dx} + q\frac{d}{dy}\right)\left(p\frac{d^2\phi}{dx\,dy} + q\frac{d^2\phi}{dy^2}\right);
\end{aligned}$$

therefore

$$v = 2\left(p^2\frac{d^2\phi}{dx^2} + 2pq\frac{d^2\phi}{dx\,dy} + q^2\frac{d^2\phi}{dy^2}\right) + \mathrm{V}(u),$$

where thus far V is any arbitrary function of u alone.

48. As yet no account has been taken of derivatives with regard to u; but the whole of the equations are subsidiary to the construction of an integral of the original equation.

The part of the value of v given by V (u) can be dropped, as it has been considered in the earlier stage, or it can be regarded as absorbed in the other part of the value, all that is necessary for this purpose being that the function ϕ should have a different form.

Assuming this done, the function ϕ must be such as, first, to satisfy the equation

$$\frac{dv}{du} = n\frac{dz}{du} = n\Delta,$$

with the preceding notation. Now

$$\frac{dv}{du} = 2\left(p\frac{d}{dx} + q\frac{d}{dy}\right)\left\{p\frac{d^2\phi}{dx\,du} + q\frac{d^2\phi}{dy\,du} + 2\left(p'\frac{d}{dx} + q'\frac{d}{dy}\right)\phi\right\};$$

also

$$n = \left(p\frac{d}{dx} + q\frac{d}{dy}\right)\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right);$$

and, as

$$\Delta = 1 + xp' + yq',$$

we have

$$\left(p\frac{d}{dx} + q\frac{d}{dy}\right)\Delta = 0:$$

that is, Δ behaves as a constant with regard to the operator $p\frac{d}{dx} + q\frac{d}{dy}$. The required equation will therefore be satisfied if

$$p\frac{d^2\phi}{dx\,du} + q\frac{d^2\phi}{dy\,du} + 2p'\frac{d\phi}{dx} + 2q'\frac{d\phi}{dy} = \tfrac{1}{2}\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right)\Delta.$$

But it is necessary that the other three equations

$$\frac{1}{g}\frac{dl}{du} = \frac{1}{f}, \quad \frac{dm}{du} = \frac{1}{c}, \quad \frac{dn}{du} = \frac{dz}{du},$$

should also be satisfied; that they lead to no new condition for ϕ can be seen as follows. We have

$$\begin{aligned}
&\left(p\frac{d^2}{dx^2} + 2q\frac{d^2}{dx\,dy} - p\frac{d^2}{dy^2}\right)\left\{\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right)\Delta\right\}\\
&\quad = \Delta\left(p,\ q,\ -p\right)\!\!\left(\frac{d}{dx},\ \frac{d}{dy}\right)^2\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right)\\
&\qquad + 2p\left\{p'\left(\frac{d^3\phi}{dx^3} + \frac{d^3\phi}{dx\,dy^2}\right)\right\}\\
&\qquad + 2q\left\{q'\left(\frac{d^3\phi}{dx^3} + \frac{d^3\phi}{dx\,dy^2}\right) + p'\left(\frac{d^3\phi}{dx^2\,dy} + \frac{d^3\phi}{dy^3}\right)\right\}\\
&\qquad - 2p\left\{q'\left(\frac{d^3\phi}{dx^2\,dy} + \frac{d^3\phi}{dy^3}\right)\right\}\\
&\quad = \Delta 2g + 2\left(p'q - pq'\right)\left(\frac{d^3\phi}{dx^2\,dy} + \frac{d^3\phi}{dy^3}\right);
\end{aligned}$$

so that

$$\left(p\frac{d^2}{dx^2} + 2q\frac{d^2}{dx\,dy} - p\frac{d^2}{dy^2}\right)\left\{\tfrac{1}{2}\left(\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2}\right)\Delta\right\} = \Delta g + \left(p'q - pq'\right)\left(\frac{d^3\phi}{dx^2\,dy} + \frac{d^3\phi}{dy^3}\right).$$

On account of the equation satisfied by ϕ, this is equal to

$$\left(p\frac{d^2}{dx^2} + 2q\frac{d^2}{dx\,dy} - p\frac{d^2}{dy^2}\right)\left\{\left(p\frac{d}{dx} + q\frac{d}{dy}\right)\frac{d\phi}{du} + 2\left(p'\frac{d}{dx} + q'\frac{d}{dy}\right)\phi\right\},$$

which is equal to

$$\begin{aligned}&\frac{d}{du}\left[\left(p\frac{d}{dx}+q\frac{d}{dy}\right)\left(p\frac{d^2}{dx^2}+2q\frac{d^2}{dx\,dy}-p\frac{d^2}{dy^2}\right)\phi\right]\\ &\quad+\left(p'\frac{d}{dx}+q'\frac{d}{dy}\right)\left(p\frac{d^2}{dx^2}+2q\frac{d^2}{dx\,dy}-p\frac{d^2}{dy^2}\right)\phi\\ &\quad-\left(p\frac{d}{dx}+q\frac{d}{dy}\right)\left(p'\frac{d^2}{dx^2}+2q'\frac{d^2}{dx\,dy}-p'\frac{d^2}{dy^2}\right)\phi\\ &=\frac{dl}{du}+(p'q-pq')\left(\frac{d^3\phi}{dx^2\,dy}+\frac{d^3\phi}{dy^3}\right);\end{aligned}$$

and therefore

$$\frac{dl}{du}=\Delta g=g\frac{dz}{du},$$

in virtue of the equation

$$p\frac{d^2\phi}{dx\,du}+q\frac{d^2\phi}{dx\,du}+2\left(p'\frac{d\phi}{dx}+q'\frac{d\phi}{dy}\right)=\tfrac{1}{2}\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}\right)\Delta.$$

Similarly it can be verified that

$$\frac{dm}{du}=f\frac{dz}{du},\qquad \frac{dn}{du}=c\frac{dz}{du},$$

in virtue of this equation; therefore it is the only condition to be imposed upon ϕ in order that all the equations may be satisfied.

49. Solutions of the equation $a+b+c=0$ have been given in §§ 31–38; it is not difficult to verify that they are included in the preceding general form. Thus, taking the solution

$$v=f(u)$$

and assuming that ϕ is, as in § 48, the function through which the term $\mathrm{V}(u)$ in v is absorbed, it is first necessary to determine ϕ so that

$$p^2\frac{d^2\phi}{dx^2}+2pq\frac{d^2\phi}{dxdy}+q^2\frac{d^2\phi}{dy^2}=\tfrac{1}{2}f(u).$$

This equation is easily solved; and we have

$$\phi=\tfrac{1}{4}\frac{x^2}{p^2}f(u)+\mathrm{G}(u,\eta)+x\mathrm{H}(u,\eta),$$

where G and H are arbitrary functions, and η denotes $p'x+q'y$, as before. We have

$$\begin{aligned}\frac{d^2\phi}{dx^2}&=\tfrac{1}{2}\frac{1}{p^2}f(u)+p'^2\left(\frac{\partial^2\mathrm{G}}{\partial\eta^2}+x\frac{\partial^2\mathrm{H}}{\partial\eta^2}\right)+2p'\frac{\partial\mathrm{H}}{\partial\eta},\\ \frac{d^2\phi}{dx\,dy}&=\qquad\qquad\quad p'q'\left(\frac{\partial^2\mathrm{G}}{\partial\eta^2}+x\frac{\partial^2\mathrm{H}}{\partial\eta^2}\right)+q'\frac{\partial\mathrm{H}}{\partial\eta},\\ \frac{d^2\phi}{dy^2}&=\qquad\qquad\quad q'^2\left(\frac{\partial^2\mathrm{G}}{\partial\eta^2}+x\frac{\partial^2\mathrm{H}}{\partial\eta^2}\right);\end{aligned}$$

so that

$$\psi = \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = \tfrac{1}{2}\frac{f(u)}{p^2} - \theta^2\left(\frac{\partial^2 \mathrm{G}}{d\eta^2} + x\,\frac{\partial^2 \mathrm{H}}{\partial\eta^2}\right),$$

$$\chi = \Big(-q,\ p,\ q\Big)\Big(\frac{d}{dx},\ \frac{d}{dy}\Big)^2 \phi = -\tfrac{1}{2}\frac{qf(u)}{p^2} + q\theta^2\left(\frac{\partial^2 \mathrm{G}}{\partial\eta^2} + x\,\frac{\partial^2 \mathrm{H}}{\partial\eta^2}\right) + 2\,(pq' - p'q)\,\frac{\partial \mathrm{H}}{\partial\eta},$$

$$\theta = \Big(p,\ q,\ -p\Big)\Big(\frac{d}{dx},\ \frac{d}{dy}\Big)^2 \phi = \tfrac{1}{2}\frac{f(u)}{p} + p\theta^2\left(\frac{\partial^2 \mathrm{G}}{\partial\eta^2} + x\,\frac{\partial^2 \mathrm{H}}{\partial\eta^2}\right).$$

Hence

$$l = \left(p\,\frac{d}{dx} + q\,\frac{d}{dy}\right)\theta = \quad p^2\theta^2\,\frac{\partial^2 \mathrm{H}}{\partial\eta^2},$$

$$m = \left(p\,\frac{d}{dx} + q\,\frac{d}{dy}\right)\chi = \quad pq\theta^2\,\frac{\partial^2 \mathrm{H}}{\partial\eta^2},$$

$$n = \left(p\,\frac{d}{dx} + q\,\frac{d}{dy}\right)\psi = -\,p\theta^2\,\frac{\partial^2 \mathrm{H}}{\partial\eta^2}.$$

But ϕ (or, what is the same thing in effect, G and H) must be such as to give

$$\frac{dv}{du} = n\,\frac{dz}{du},$$

so that

$$n = \frac{f'(u)}{\Delta};$$

consequently we have

$$\frac{l}{-p} = \frac{m}{-q} = \frac{n}{1} = \frac{f'(u)}{\Delta},$$

which are the proper values of l, m, n, connected with $v = f(u)$.

Similarly for the quantities a, b, c, f, g, h.

And if the value of H be required, it is determined by the equation

$$-\,p\theta^2\,\frac{\partial^2 \mathrm{H}}{\partial\eta^2} = n = \frac{f'(u)}{\Delta} = \frac{f'(u)}{1+\eta},$$

the integration of which is immediate.

Application to $\nabla^2 v + \kappa^2 v = 0$.

50. The integrable combinations of the subsidiary system in the case of the potential-equation seem fortuitously obtained. As one other illustration, added chiefly to show that the subsidiary system can be used in the mode indicated in § 39 to express all the variable quantities in terms of a single quantity, we consider the equation $\nabla^2 v + \kappa^2 v = 0$, which is

$$a + b + c + \kappa^2 v = 0,$$

in the notation of the present paper.

We have

$$A = B = C = 1, \qquad F = G = H = 0,$$
$$X = \kappa^2 l, \qquad Y = \kappa^2 m, \qquad Z = \kappa^2 n.$$

Thus the characteristic invariant equation is

$$p^2 + q^2 + 1 = 0,$$

from which we have, as before,

$$z = u + xp(u) + yq(u),$$

so that p and q are parametric for differentiations with regard to x and y. The three subsidiary equations are

$$\left.\begin{aligned} \kappa^2 l + \frac{da}{dx} - p\frac{dg}{dx} + \frac{dh}{dy} - q\frac{dg}{dy} &= 0 \\ \kappa^2 m + \frac{dh}{dx} - p\frac{df}{dx} + \frac{db}{dy} - q\frac{df}{dy} &= 0 \\ \kappa^2 n + \frac{dg}{dx} - p\frac{dc}{dx} + \frac{df}{dy} - q\frac{dc}{dy} &= 0 \end{aligned}\right\};$$

and we have the equations of definition and derivation as before.

51. One simple mode of proceeding is to use the six equations involving derivatives of l, m, n, with regard to x and y—they are equivalent to five independent equations—in order to express a, b, f, g, h, in terms of c and those derivatives. These being obtained in the form

$$a = \frac{dl}{dx} - p\frac{dn}{dx} + p^2 c, \qquad g = \frac{dn}{dx} - pc,$$
$$b = \frac{dm}{dy} - q\frac{dm}{dy} + q^2 c, \qquad f = \frac{dn}{dy} - qc,$$
$$h = \frac{dm}{dx} - p\frac{dn}{dy} + pqc = \frac{dl}{dy} - q\frac{dn}{dx} + pqc,$$

we substitute them in the subsidiary equations. The third of the latter then becomes

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2\right) n = 2\left(p\frac{d}{dx} + q\frac{d}{dy}\right) c;$$

the second of them, on using the first of the two values for h and also the relation between n and c just deduced, gives

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2\right) m = \left(-q\frac{d^2}{dx^2} + 2p\frac{d^2}{dx\,dy} + q\frac{d^2}{dy^2} - q\kappa^2\right) n;$$

and the first of them, on using the second of the two values for h and also the relation between n and c, gives

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2\right) l = \left(p\frac{d^2}{dx^2} + 2q\frac{d^2}{dx\,dy} - p\frac{d^2}{dy^2} - p\kappa^2\right) n.$$

Let the various operators be denoted by δ_1, δ_2, δ_3, say

$$\left.\begin{aligned} \delta_1 &= \quad p\frac{d^2}{dx^2} + 2q\frac{d^2}{dx\,dy} - p\frac{d^2}{dy^2} - p\kappa^2 \\ \delta_2 &= -q\frac{d^2}{dx^2} + 2p\frac{d^2}{dx\,dy} + q\frac{d^2}{dy^2} - q\kappa^2 \\ \delta_3 &= \quad\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \kappa^2 \end{aligned}\right\};$$

moreover, let

$$\delta = p\frac{d}{dx} + q\frac{d}{dy}$$

be another operator. Then from the relations between m and n, l and n, it follows that a function Θ exists such that

$$l = \delta_1\Theta, \qquad m = \delta_2\Theta, \qquad n = \delta_3\Theta\,;$$

and then, as

$$\left.\begin{aligned} \frac{dv}{dx} &= l + np = 2\left(p\frac{d^2}{dx^2} + q\frac{d^2}{dx\,dy}\right)\Theta \\ \frac{dv}{dy} &= m + nq = 2\left(p\frac{d^2}{dx\,dy} + q\frac{d^2}{dy^2}\right)\Theta \end{aligned}\right\},$$

we have

$$v = 2\left(p\frac{d}{dx} + q\frac{d}{dy}\right)\Theta = 2\delta\Theta.$$

From the relation between n and c, we have

$$2\delta c = \delta_3 n = \delta_3{}^2\Theta,$$

or, if $\Theta = \delta w$, then

$$2c = \delta_3{}^2 w.$$

We thus have, from the foregoing relations, the results

$$\left.\begin{aligned} v &= 2\delta^2 w \\ l &= \delta\delta_1 w \\ m &= \delta\delta_2 w \\ n &= \delta\delta_3 w \\ c &= \tfrac{1}{2}\delta_3{}^2 w \end{aligned}\right\}.$$

The values of a, b, f, g, h, can be obtained by using these values of l, m, n, c, and we find

$$\left.\begin{aligned} a &= \tfrac{1}{2}\delta_1{}^2 w \\ h &= \tfrac{1}{2}\delta_1\delta_2 w \\ b &= \tfrac{1}{2}\delta_2{}^2 w \\ g &= \tfrac{1}{2}\delta_1\delta_3 w \\ f &= \tfrac{1}{2}\delta_2\delta_3 w \end{aligned}\right\} .$$

All these forms correspond to the respective expressions obtained in § 47 for the potential-equation.

52. Account must now be taken of derivatives with regard to u, and the equations to be satisfied are

$$\frac{1}{n}\frac{dv}{du} = \frac{1}{g}\frac{dl}{du} = \frac{1}{f}\frac{dm}{du} = \frac{1}{c}\frac{dn}{du} = \frac{dz}{du} .$$

Taking the first, we have

$$\frac{dz}{du} = 1 + xp' + yq' = \Delta,$$

so that the equation is to be

$$\frac{dv}{du} = n\Delta = \Delta\delta\delta_3 w.$$

Now, denoting

$$p'\frac{d}{dx} + q'\frac{d}{dy} \text{ by } \delta',$$

and so for δ_1', δ_2'—viz., δ_1' is the same as δ_1 except that p' and q' replace p and q, and so for δ_2'—we have

$$\begin{aligned} \frac{dv}{du} &= \frac{d}{du}(2\delta^2 w) \\ &= 2\delta^2\frac{dw}{du} + 4\delta\delta' w ; \end{aligned}$$

and therefore

$$2\delta^2\frac{dw}{du} + 4\delta\delta' w = \Delta\delta\delta_3 w.$$

Now

$$\delta\Delta = pp' + qq' = 0,$$

so that

$$\delta(\Delta\delta_3 w) = \Delta\delta\delta_3 w.$$

Hence the equation in w will be satisfied if

$$2\delta\frac{dw}{du} + 4\delta' w = \Delta\delta_3 w$$

the condition that we may have

$$\frac{dv}{du} = n\frac{dz}{du}.$$

But now, operating on this equation with δ_1, we have

$$\delta_1\left[2\delta\frac{dw}{du}+4\delta' w\right]=\delta_1(\Delta\delta_3 w).$$

The left-hand side is

$$2\delta\delta_1\frac{dw}{du}+4\delta'\delta_1 w=2\delta\delta_1\frac{dw}{du}+2\delta'\delta_1 w+2\delta_1'\delta w+2(\delta'\delta_1-\delta_1'\delta)\,\omega$$

$$=\frac{d}{du}\{2\delta\delta_1 w\}+2(\delta'\delta_1-\delta_1'\delta)\,w.$$

The first term is $2\dfrac{dl}{du}$; the second, on reduction, gives

$$2(qp'-pq')\frac{d}{dy}\delta_3 w;$$

and therefore the left-hand side is

$$2\frac{dl}{du}+2(qp'-pq')\frac{d}{dy}\delta_3 w.$$

The right-hand side, viz., $\delta_1(\Delta\delta_3 w)$, becomes on expansion and reduction

$$\Delta\delta_1\delta_3 w+2(p'q-pq')\frac{d}{dy}\delta_3 w=\Delta 2g+2(p'q-pq')\frac{d}{dy}\delta_3 w.$$

Hence, in virtue of the above equation, we have

$$\frac{dl}{du}=g\Delta=g\frac{dz}{du}.$$

Similarly we find

$$\frac{dm}{du}=f\frac{dz}{du},\qquad \frac{dn}{du}=c\frac{dz}{du};$$

and therefore the equation determining w is

$$\delta\frac{dw}{du}+2\delta' w=\tfrac{1}{2}\Delta\delta_3 w,$$

that is,

$$p\frac{d^2w}{dx\,du}+q\frac{d^2w}{dy\,du}+2p'\frac{dw}{dx}+2q'\frac{dw}{dy}=\tfrac{1}{2}(1+xp'+yq')\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\kappa^2 w\right).$$

When a solution of this equation is obtained, a solution of the equation for v can be constructed, and conversely. In particular, the identification of any solution of the v-equation, as being included in the general solution, can be made as in former cases.

[NOTE.—Added 17th March, 1898. As indicated in the introductory remarks, the theory given in this paper is applicable when the number of independent variables is greater than two, and when the order of the differential equation is higher than the first; its application is not restricted to the case of an equation of the second order in three independent variables.

A brief sketch of the theory for an equation of order m in n independent variables is prefixed to a paper* which contains the integration of some differential equations of types represented by

$$\left(\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + \dots\dots + x_{n-1}\frac{\partial}{\partial x_n}\right)^m \mathrm{U} = 0,$$

when the iteration of the differential operator is purely symbolical.]

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* "On some Differential Equations in the Theory of Symmetrical Algebra," Camb. Phil. Trans., vol. xvi. (1898), pp. 291–325.